GLOBAL ARTIFICIAL INTELLIGENCE CHAMPIONSHIPS (GAIC) MATH 2024

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Problem 1. *Let $S = \{1, 2, \cdots 2024\}$, if the set of any n pairwise prime numbers in S has at least one prime number, the minimum value of n is __________.*

Answer: 16
Reasoning: Taking the 15 numbers 1, 22, 32, ..., 432 violates the condition. Furthermore, since S does not contain any non-prime numbers with a minimum prime factor of at least 47, there are only 14 types of non-prime numbers in S, excluding 1. Applying the Pigeonhole Principle, we conclude that $n = 16$.

Problem 2. *Let $A_l = (4l+1)(4l+2)\cdots(4(5^5+1)l)$. Given a positive integer l such that $5^{25l} \mid A_l$ and $5^{25l+1} \nmid A_l$, the minimum value of l satisfying these conditions is __________.*

Answer: 3906
Reasoning: Let $n = 4l$.

Then $A_l = (5^5 l)! C^n_{(5^5+1)n}$, where $\nu_5((5^5 l)!) = 5^4 n + 5^3 n + \cdots + n + \nu_5(n!) = \frac{5^5-1}{4} n + \nu_5(n!)$.

Thus, we need $\nu_5\left(C^n_{(s^5+1)n}\right) = \frac{n}{4} - \nu_5(n!)$.

By Kummer's Theorem, this means in base 5, when adding n and $5^5 n$, there are $\frac{n}{4} - \nu_5(n!)$ carries.

Notice that $\frac{n}{4} - \nu_5(n!) > \frac{n}{4} - \left(\frac{n}{5} + \frac{n}{25} + \cdots\right) = 0$,

which implies there must be carries when adding n and $5^5 n$. Thus, n must have at least 6 digits in base 5.

Suppose $n = a_5 a_4 \cdots a_0$. The number of carries when adding n and $5^5 n$ is the same as the number of carries when adding a_5 and $\overline{a_5 a_4 \cdots a_0}$.

Since

$$\begin{aligned}\frac{n}{4}-v_5(n!)&=\left(\frac{n}{5}+\frac{n}{25}+\cdots\right)-\left(\left[\frac{n}{5}\right]+\left[\frac{n}{25}\right]+\cdots\right)\\&=\left\{\frac{n}{5}\right\}+\left\{\frac{n}{25}\right\}+\cdots\\&=\frac{a_0}{5}+\frac{5a_1+a_0}{25}+\frac{25a_2+5a_1+a_0}{125}+\cdots\\&=\frac{a_5+a_4+\cdots+a_0}{4}\\&\geqslant\frac{a_5+a_0}{4}\geqslant\frac{5}{4}>1\end{aligned}$$

This indicates that in the quintile system a_5 and $\overline{a_5a_4\cdots a_0}$ are carried at least 2 times, then $a_1=4$. While,

$$\frac{a_5+a_4+\cdots+a_0}{4}\geqslant\frac{a_5+a_1+a_0}{4}\geqslant\frac{9}{4}>2,$$

which implies $a_2=4$. Continuing this process, we find that $a_1=a_2=\cdots=a_5=4$.

And then $\frac{a_5+a_4+\cdots+a_0}{4}\in\mathbf{Z}$, we get $a_0=4$.

Obviously, such n is indeed satisfied the requirements. Therefore, the minimum value of l that satisfies the condition is $\frac{5^6-1}{4}=3906$

Finally, note that the number of carries when adding $5n$ to 5^6n is the same as the number of carries when adding $5n$ to 5^5n, which means if l satisfies the conditions, then $5l$ also satisfies the conditions, implying that there are infinitely many values of l satisfying the conditions.

Problem 3. *Sasha collects coins and stickers, with fewer coins than stickers, but at least 1 coin. Sasha chooses a positive number $t>1$ (not necessarily an integer). If he increases the number of coins by a factor of t, then he will have a total of 100 items in his collection. If he increases the number of stickers by a factor of t, then he will have a total of 101 items in his collection. If Sasha originally had more than 50 stickers, then he originally had __________ stickers.*

Answer: 66

Reasoning: Let m and n be the number of coins and stickers Sasha originally had, respectively.

According to the problem, we have:

$$mt+n=100, \tag{1}$$
$$m+nt=101. \tag{2}$$

From (2)−(1), we can learn that
$(n-m)(t-1)=1\Longrightarrow t=1+\frac{1}{n-m}$.
From (1)+(2), we can learn that
$(n+m)(t+1)=201\Longrightarrow t=\frac{201}{m+n}-1$.
Let $a=n-m$ and $b=n+m$.
Since $n>m$, we have $a>0$.

Comparing the two different expressions in terms of t, we get:
$1+\frac{1}{n-m}=\frac{201}{m+n}-1 \Leftrightarrow 1+\frac{1}{a}=\frac{201}{b}-1 \Leftrightarrow \frac{2a+1}{a}=\frac{201}{b}$.

Because $2a+1$ and a are coprime, $\frac{2n+1}{a}$ is in lowest terms, which implies that 201 is divisible by $2a+1$. Since $201 = 3 \times 67$, it has only four positive divisors: 1, 3, 67, and 201.

Since $2a+1 > 1$, there are three possible cases:

1. $2a+1 = 3$.

   Then $a = 1 \Rightarrow \frac{201}{b} = 3 \Rightarrow b = 67$.

   Hence, $m = \frac{1}{2}(b-a) = 33$,

   $n = \frac{1}{2}(a+b) = 34$,

   and $t = 2$,

   which does not satisfy the condition.

2. $2a+1 = 67$.

   Then $a = 33 \Rightarrow \frac{201}{b} = \frac{67}{33} \Rightarrow b = 99$.

   Hence, $m = \frac{1}{2}(b-a) = 33$,

   $n = \frac{1}{2}(a+b) = 66$,

   and $t = \frac{34}{33}$.

   It can be easily verified that this case satisfies the condition.

3. $2a+1 = 201$.

   Then $a = 100 \Rightarrow \frac{201}{b} = \frac{201}{100} \Rightarrow b = 100$.

   Hence, $m = \frac{1}{2}(b-a) = 0$,

   and $n = \frac{1}{2}(a+b) = 100$.

   Since the number of coins cannot be 0, this case does not satisfy the condition.

Problem 4. *Let n be a positive integer. An integer k is called a "fan" of n if and only if $0 \leqslant k \leqslant n-1$ and there exist integers $x,\ y,\ z \in \mathbf{Z}$ such that $x^2+y^2+z^2 \equiv 0 (\bmod n)$ and $xyz \equiv k (\bmod n)$. Let $f(n)$ denote the number of fans of n. Then f(2020) = ________.*

Answer: 101
Reasoning: For a fan k of 2020, since there exists $x^2+y^2+z^2 \equiv 0 (\bmod 2020)$, particularly, we have $x^2+y^2+z^2 \equiv 0 (\bmod 4)$.

Thus, $x^2 \equiv 0$ or $1 (\bmod 4)$, which implies $x,\ y,\ z$ are all even.

Therefore, $k \equiv xyz \equiv 0 (\bmod 4)$.

Also, $x^2+y^2+z^2 \equiv 0 (\bmod 5)$, and $x^2 \equiv 0,\ 1,\ 4 (\bmod 5)$, so there must be a number among $x,\ y,\ z$ that is a multiple of 5.

Hence, $k \equiv 0 (\bmod 5)$.

Therefore, a fan k of 2020 must be a multiple of 20.

Next, we prove that all multiples of 20 are fans of 2020.

Since $x^2+y^2+z^2 \equiv 0 (\bmod 101)$ has solutions, let $x = a$, $y = 6a$, $z = 8a$, then any $k \equiv xyz \equiv 48a^3 (\bmod 101)$ is fan of 101.

If there exist $i \neq j$ such that $48i^3 \equiv 48j^3 (\bmod 101)$ $(0 \leqslant i < j \leqslant 100)$, then $(i-j)(i^2+ij+j^2) \equiv 0 (\bmod 101)$.

Since $i - j \neq 0 (\bmod 101)$, we have $i^2+ij+j^2 \equiv 0 (\bmod 101)$.

Thus, $(2i+j)^2 \equiv -3j^2 (\bmod 101)$, implying that -3 is a quadratic residue modulo 101.

But from the law of quadratic reciprocity, $\left(\frac{3}{101}\right)\left(\frac{101}{3}\right) = (-1)^{\frac{3-1}{2}\cdot\frac{101-1}{2}} = 1$, and $\left(\frac{101}{3}\right) = -1$,

so $\left(\frac{3}{101}\right) = -1$.

Furthermore, $\left(\frac{-3}{101}\right) = \left(\frac{3}{101}\right)\left(\frac{-1}{101}\right)$.

Since $100 \equiv 10^2 \equiv -1 (\bmod 101)$, we have $\left(\frac{-1}{101}\right) = 1$, which implies $\left(\frac{-3}{101}\right) = -1$, contradicting our assumption.

This indicates that when i traverses the complete system modulo 101, $48i^3$ also traverses the complete system modulo 101. Since $0 \leqslant k \leqslant 2019$, we conclude that $f(2020) = 101$.

Problem 5. *Four positive integers satisfy $a^3 = b^2$, $c^5 = d^4$, and $c - a = 77$. Then, $d - b =$ __________.*

Answer: 235
Reasoning: Given the conditions, we can assume $a^3 = b^2 = x^6$ and $c^5 = d^4 = y^{20}$, which yields $y^4 - x^2 = (y^2 - x)(y^2 + x) = 77$. Hence, we find that $y = 3,\ x = 2,\ d - b = 243 - 8 = 235$.

Problem 6. *The smallest n such that both $3n+1$ and $5n+1$ are perfect squares is* __________.

Answer: 16
Reasoning: It can be easily verified.

Problem 7. *Find the largest positive integer n such that the product of the numbers $n, n+1, n+2, \cdots, n+100$ is divisible by the square of one of these numbers.*

Answer: 100!
Reasoning: When n=100!, $\frac{n(n+1)(n+2)...(n+100)}{n^2} = \binom{n+100}{100}$is an integer.

If $n > 100!$, Let the product be divisible by the square of $n+k$, then: $n(n+1)(n+2)...(n+k-1)(n+k+1)(n+k+2)\cdots(n+100) \equiv 0 (\bmod n+k)$, namely, $-1^k k!(100-k)! \equiv 0 (\bmod n+k)$.

But by $n > 100!$, $-1^k k!(100-k)! < n+k$, and $-1^k k!(100-k)!$ non-zero, resulting in contradiction.

So the maximum n is 100!

Problem 8. *Given a positive integer x with m digits in its decimal representation, and let x^3 have n digits. Which of the following options cannot be the value of $m+n$?*

A) 2022	*B)* 2023
C) 2024	*D)* 2025

Answer: D
Reasoning: Given that $10^{m-1} \leq x < 10^m$, then $10^{3m-3} \leq x^3 < 10^{3m}$. Hence, n can take values $3m-2$, $3m-1$, or $3m$. Thus, $m+n$ cannot be congruent to 1 modulo 4. Therefore, option D is chosen.

Problem 9. *Positive integers a, b, and c satisfy $a > b > c > 1$, and also satisfy $abc \mid (ab-1)(bc-1)(ca-1)$. There are __________ possible sets of (a,b,c).*

Answer: 1
Reasoning: The original statement is equivalent to $abc \mid ab+bc+ca-1$. It's evident that $c<3$, so $c=2$.

Then, $2ab < ab+2a+2b$ implies $b<4$, so $b=3$.

Substituting back, we find $a=5$.

Problem 10. *There are __________ sets of positive integers $a \leq b \leq c$ such that $ab-c$, $bc-a$, and $ca-b$ are all powers of 2.*

Answer: 4
Reasoning: Only the sets $(2,2,2)$,$(2,2,3)$,$(2,6,11)$ and $(3,5,7)$ satisfy the conditions.

Problem 11. *Define the function $f(x) = [x[x]]$, where $[x]$ represents the largest integer not exceeding x. For example, $[-2.5] = -3$.For a positive integer n, let a_n be the number of elements in the range set of $f(x)$ when $x \in [0,n)$. Then the minimum value of $\frac{a_n+90}{n}$ is __________.*

Answer: 13
Reasoning: When $x \in [0,1)$, only one value satisfies $f(x)=0$, for the positive integer k, when $x \in [k,k+1)$, $x[x]=kx$, then $f(x)$ has a total of k values on $[k,k+1)$, and obviously the values on different intervals are not equal to each other, then we have:

$$a_n = 1+1+2+...+(n-1) = \frac{n^2-n+2}{2}$$

So $\frac{a_n+90}{n} = \frac{\frac{n^2-n+2}{2}+90}{n} = \frac{n}{2}+\frac{91}{n}-\frac{1}{2}$, by the mean value inequality positive integer n should be around $\sqrt{2\times 91}$. Substituting $n=13$ and $n=14$ both yield a result of 13.

Problem 12. *If a positive integer's sum of all its positive divisors is twice the number itself, then it is called a perfect number. If a positive integer n satisfies both $n-1$ and $\frac{n(n+1)}{2}$ being perfect numbers, then $n=$ __________.*

Answer: 7
Reasoning: Here we need to use a result from Euler:

n is an even perfect number $\Leftrightarrow$ there exists a prime p such that 2^p-1 is prime, and $n = 2^{p-1}(2^p-1)$.

Now let's use this to solve the problem.

Case 1: n is odd. Then $n-1$ is an even perfect number. We can write $n-1 = 2^{p-1}(2^p-1)$, where both p and 2^p-1 are primes. In this case,

$$\frac{n\left(n+1\right)}{2} = \frac{1}{2}\left(2^{p-1}\left(2^p-1\right)+1\right)\left(2^{p-1}\left(2^p-1\right)+2\right) = \left(2^{p-1}\left(2^p-1\right)+1\right)\left(2^{p-2}\left(2^p-1\right)+1\right).$$

When $p=2$, $n=7$, $\frac{n(n+1)}{2}=28$, in this case, $n-1$ and $\frac{n(n+1)}{2}$ are both perfect numbers.

When $p \geq 3$,let $N=\frac{n(n+1)}{2}$, then N is odd, and

$$\frac{n+1}{2}=4^{p-1}-2^{p-2}+1=(3+1)^{p-1}-(3-1)^{p-2}+1,$$

We know by the binomial theorem that $\equiv 3\times(p-1)-(p-2)\times 3+1+1+1\equiv 6(\text{mod}9)$.

Thus $3 \mid N$, but $3^2 \nmid N$, can set $N=3k, 3\times k$, in this case, $\sigma(N)=\sigma(3)$, $\sigma(k)=4\sigma(k)$, but $2N=2(\text{mod}4)$, so $\sigma(N)\neq 2N$, so $\frac{n(n+1)}{2}$ is not perfect.

Case 2: n is an even number, if $4 \mid n$, then $n-1=-1(\text{mod}4) \Rightarrow n-1$ not a perfect square number, at this point for any $d \mid n-1$, we can know that d and $\frac{n-1}{d}$ one mod 4 remains -1, and the other mod4 remains 1 from $d\times\frac{n-1}{d}=n-1\equiv -1(\text{mod}4)$, leading to $d+\frac{n-1}{d}\equiv 0(\text{mod}4)$, and $4 \mid \sigma(n-1)$, but $2(n-1)=2(\text{mod}4)$, so $n-1$ is not perfect.

So, $4 \nmid n$, then, can be set $n=4k+2$, now $N=\frac{n(n+1)}{2}=(2k+1)(4k+3)$ is odd. Because of $(2k+1, 4k+3)=1$, so $\sigma(N)=\sigma(2k+1)\,\sigma(4k+3)$.

Same as above knowable $4 \mid \sigma(4k+3)$, if $\sigma(N)=2N$, we can learn that $4 \mid 2N \Rightarrow 2 \mid N$, this is a contradiction.

To sum up, there is only one n that meets the condition, that is, $n=7$.

Problem 13. *Try to find all prime numbers with the shape p^p+1(p is a natural number) that have no more than 19 digits and what the sum of these prime numbers is.*

Answer: 264

Reasoning: Obviously $p<19$.

If p is odd, then p^p+1 is divisible by 2, so $p^p+1=2$ is prime only if $p=1$.

If p has an odd factor, let $p=mk$, of which m is odd, then

$p^p+1=p'+1=\left(p^k\right)^m+1$, at this time, $p^k+1 \mid p^p+1$.

Thus $p^{\rho+1}$is not a prime number.

Thus, p can only be an even number less than 19, with only even factors, i.e. $p=$2,4, 8, 16. If $p=16$, then

$$16^{18}=2^{64}=\left(2^{10}\right)^6\cdot 16>1000^6\cdot 16=16\cdot 10^{18},$$

Then $16^{16}+1$is more than 19 digits.

If $p=8$, then

$$8^8+1=2^{24}+1=\left(2^8\right)^3+1=\left(2^8+1\right)\left(2^{16}-2^8+1\right)\text{ is a composite number.}$$

If $p=4$, then $4^4+1=257$ is prime.

If $p=2$, then $2^2+1=5$ is prime.

So the prime numbers are 2, 5, 257, and their sum is 264.

Problem 14. *For any positive integer q_0, consider a sequence $q_1, q_2, \cdots, q_n$ defined by $q_i=(q_{i-1}-1)^3+3\,(i=1,2,\cdots,n)$. If every $q_i\,(i=1,2,\cdots,n)$ is a power of prime, then the maximum possible value of n is __________.*

Answer: 2

Reasoning: Since $m^3-m=m(m-1)(m+1)\equiv 0 \pmod 3$, we have $q_i = (q_{i-1} - 1)^3 + 3 \equiv (q_{i-1} - 1)^3 \equiv q_{i-1} - 1 \pmod 3$.

Therefore, among q_1, q_2, and q_3, at least one must be divisible by 3, and this number should be a power of 3.

If $3 \mid (q-1)^3 + 3$, then $3 \mid (q-1)^3$, implying $3 \mid q - 1$.

Thus, $3^3 \mid (q-1)^3$.

Since $3 \mid (q-1)^3 + 3$, it follows that $(q-1)^3 + 3$ is a multiple of 3 only when $q_i = 1$, and this occurs only when $i = 0$.

However, when $q_0 = 1$, we get $q_1 = 3$, $q_2 = 11$, and $q_3 = 1003 = 17 \times 59$. Therefore, the maximum value of n is 2.

Problem 15. *The first digit before the decimal point in the decimal representation of* $(\sqrt{2} + \sqrt{5})^{2000}$ *is* __________ *and after the decimal point is* __________.

Answer: 1,9

Reasoning: $\left(\sqrt{2} + \sqrt{5}\right)^{2000} = \left(7 + 2\sqrt{10}\right)^{1000}$.

Let $a_n = \left(7 + 2\sqrt{10}\right)^n + \left(7 - 2\sqrt{10}\right)^n$, then $a_0 = 2, a_1 = 14$.

Note that a_n is a second-order recursive sequence whose characteristic equation is $\left[t - \left(7 + 2\sqrt{10}\right)\right]\left[t - \left(7 - 2\sqrt{10}\right)\right] = 0$, i.e. $t^2 - 14t + 9 = 0$.

So $a_{n+2} - 14a_{n+1} + 9a_n = 0$.

Thus, a_n is a series of integers. Calculate the remainder of the first few terms in the sequence modulo 10:

$$a_0 \equiv 2(\text{mod}10), a_1 \equiv 4(\text{mod}10), a_2 \equiv 8(\text{mod}10) .$$

$$a_3 \equiv 6(\text{mod}10), a_4 \equiv 2(\text{mod}10), a_5 \equiv 4(\text{mod}10).$$

Notice the $a_0 \equiv a_4 \,(\text{mod}10)\,, a_1 \equiv a_5 \,(\text{mod}10)$, since $\{a_n\}$ is second order recursive sequence, then $a_{n+4} \equiv \alpha_n(\text{mod}10)$.

Thus $a_{1000} \equiv a_{996} \equiv a_{992} \equiv \cdots \equiv a_0 \equiv 2(\text{mod}10)$.

Since $0 < 7 - 2\sqrt{10} < 1$, then $0 < \left(7 - 2\sqrt{10}\right)^{1000} < 1$.

So $\left[\left(7 + 2\sqrt{10}\right)^{1000}\right] = a_n - 1 \equiv 1\,(\text{mod}10)$.

Then, in the decimal representation of $\left(\sqrt{2} + \sqrt{5}\right)^{2000}$, the first digit before the decimal point is 1.

Also because $0 < 7 - 2\sqrt{10} < 0.9$, then $0 < \left(7 - 2\sqrt{10}\right)^{1000} < 0.1$.

So $\left\{7 + 2\sqrt{10^n}\right\} = 1 - \left(7 - 2\sqrt{10}\right)^n > 0.9$.

Then, in the decimal representation of $\left(\sqrt{2} + \sqrt{5}\right)^{2000}$, the first digit after the decimal point is 9.

Problem 16. *N is a 5-digit number composed of 5 different non-zero digits, and N is equal to the sum of all three digits formed by 3 different digits in these 5 digits, then the sum of all such 5-digit N is* __________.

Answer: 35964

Reasoning: Let $N = \overline{a_1a_2a_3a_4a_5}$ be the 5-digit number.

There are $P_4^2 = 12$ three-digit numbers with a_1 as the hundreds digit, $P_4^2 = 12$ three-digit numbers with a_2 as the tens digit, and $P_4^2 = 12$ three-digit numbers with a_3 as the units digit. So, according to the given condition:

$$N = \overline{a_1a_2a_3a_4a_5} = (a_1 + a_2 + a_3 + a_4 + a_5)(100 \cdot 12 + 10 \cdot 12 + 12) = 1332(a_1 + a_2 + a_3 + a_4 + a_5).$$

Since $9 \mid 1332$, it follows that $9 \mid N = \overline{a_1a_2a_3a_4a_5}$, and consequently $9 \mid (a_1 + a_2 + a_3 + a_4 + a_5)$.

Thus, N must be a multiple of $1332 \cdot 9 = 11988$.

Considering the possible values of $a_1 + a_2 + a_3 + a_4 + a_5$, we have $15 = 1+2+3+4+5 \leq a_1 + a_2 + a_3 + a_4 + a_5 \leq 9 + 8 + 7 + 6 + 5 = 35$. Therefore, $a_1 + a_2 + a_3 + a_4 + a_5$ can only be 18 or 27.

1. When $a_1 + a_2 + a_3 + a_4 + a_5 = 18$,
   $\overline{a_1a_2a_3a_4a_5} = 1332 \cdot 18 = 23976$,
   but $2 + 3 + 9 + 7 + 6 = 27 \neq 18$.

2. When $a_1 + a_2 + a_3 + a_4 + a_5 = 27$,
   $\overline{a_1a_2a_3a_4a_5} = 1332 \cdot 27 = 35964$, and $3 + 5 + 9 + 6 + 4 = 27$.
   Therefore, the required 5-digit number is only 35964.

Problem 17. *If the last three digits of a positive integer n cubed are 888, then the minimum value of n is __________.*

Answer: 192
Reasoning: If the cube of a positive integer ends in 8, then the number itself must end in 2, meaning it can be written in the form $n = 10k + 2$ (where k is a non-negative integer), and hence $n^3 = (10k + 2)^3 = 1000k^3 + 600k^2 + 120k + 8$.

The digit in the tens place of n^3 is determined by $120k$.

Since we require the tens digit of n^3 to be 8, the units digit of $12k$ should be 8, meaning the units digit of k must be 4 or 9. Therefore, we can let $k = 5m + 4$(m is a non-negative integer).

Then $n^3 = \lceil 10(5m + 4) + 2\rceil^3 = 125000m^3 + 315000m^3 + 264600m + 74088$.

To make the hundreds digit of n^3 8, we need the units digit of $2646m$ to be 8. The smallest value of m that satisfies this condition is 3.

Then, $k = 5m + 4 = 19$, and $n = 10k + 2 = 192$.

It can be verified that $n^3 = 7077888$, and its last three digits are 888.

Therefore, the minimum value of n is 192.

Problem 18. *a, b are both two-digit positive integers, $100a+b$ and $201a+b$ are both four-digit perfect squares, then $a + b =$ __________.*

Answer: 81
Reasoning: Let $100a + b = m^2$, $201a + b = n^2$, then

$101a = n^2 - m^2 = (n - m)(n + m)$, $m,\ n < 100$.

So, $n - m < 100, n + m < 200$, $101 \mid (m + n)$.

Thus, $m + n = 101$.
By substituting $a = n - m = 2n - 101$, we obtain
$201(2n - 101) + b = n^2$, i.e. $n^2 - 402n + 20301 = b \in (9, 100)$.
Verify that $n = 59, m = 101 - n = 42$.
Thus, $a = n - m = 17$, $b = n^2 - 402n + 20301 = 64$, i.e.$(a, b) = (17, 64)$.
The answer is 17+64=81.

Problem 19. *For a positive integer n, which can be uniquely expressed as the sum of the squares of 5 or fewer positive integers (where two expressions with different summation orders are considered the same, such as $3^2 + 4^2$ and $4^2 + 3^2$ are considered the same expression of 25), then the sum of all the n that satisfy the conditions is* __________.

Answer: 34
Reasoning: First, prove that for all $n \geq 17$, there are more than 2 different expressions.

Since every positive integer can be expressed as the sum of the squares of four or less positive integers (Lagrange's four-squares theorem),

there exsit non-negative integer x_i, y_i, z_i, w_i $(i = 1, 2, 3, 4)$ that satisfy

$$n - 0^2 = x_0^2 + y_0^2 + z_0^2 + w_6^2,$$

$$n - 1^2 = x_1^2 + y_1^2 + z^2 + w_1^2,$$

$$n - 2^2 = x_2^2 + y_2^2 + z_2^2 + w_2^2,$$

$$n - 3^2 = x_3^2 + y_3^2 + z_3^2 + w_5^2,$$

$$n - 4^2 = x_4^2 + y_4^2 + z_4^2 + w_4^2,$$

It follows that

$$n = x_0^2 + y_0^2 + z_0^2 + w_0^2 = 1^2 + x_1^2 + y_1^2 + z_1^2 + w_1^2 = 2^2 + x_2^2 + y_2^2 + z_2^2 + w_2^2$$
$$= 3^2 + x_3^2 + y_3^2 + z_3^2 + w_3^2 = 4^2 + x_4^2 + y_4^2 + z_4^2 + u_4^2.$$

Suppose $n \neq 1^2 + 2^2 + 3^2 + 4^2 = 30$. Then,

$$\{1, 2, 3, 4\} \neq \{x_0, y_0, z_0, w_0\}.$$

Therefore, there exists $k \in \{1, 2, 3, 4\} \setminus \{x_0, y_0, z_0, u_0\}$, and for such k, $x_0^2 + y_0^2 + z_6^2 + u_0^2$ and $k^2 + x_k^2 + y_k^2 + z_2^2 + u_k^2$ are distinct.

Because $30 = 1^2 + 2^2 + 3^2 + 4^2 = 1^2 + 2^2 + 5^2$, it suffices to consider $1 \leq n \leq 16$.

The following positive integers have two or more different expressions:

$$\begin{aligned}
&4 = 2^2 = 1^2 + 1^2 + 1^2 + 1^2 \\
&5 = 1^2 + 2^2 = 1^2 + 1^2 + 1^2 + 1^2 + 1^2 \\
&8 = 2^2 + 2^2 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2 \\
&9 = 3^2 = 1^2 + 2^2 + 2^2 \\
&10 = 1^2 + 3^2 = 1^2 + 1^2 + 2^2 + 2^2 \\
&11 = 1^2 + 1^2 + 3^2 = 1^2 + 1^2 + 1^2 + 2^2 + 2^2 \\
&12 = 1^2 + 1^2 + 1^2 + 3^2 = 2^2 + 2^2 + 2^2 \\
&13 = 1^2 + 1^2 + 1^2 + 1^2 + 3^2 = 1^2 + 2^2 + 2^2 + 2^2 \\
&14 = 1^2 + 2^2 + 3^2 = 1^2 + 1^2 + 2^2 + 2^2 + 2^2 \\
&16 = 4^2 = 2^2 + 2^2 + 2^2 + 2^2
\end{aligned}$$

However, the following six positive integers have only one unique expression:

$$\begin{aligned}
&1 = 1^2 \\
&2 = 1^2 + 1^2 \\
&3 = 1^2 + 1^2 + 1^2 \\
&6 = 1^2 + 1^2 + 2^2 \\
&7 = 1^2 + 1^2 + 1^2 + 2^2 \\
&15 = 1^2 + 1^2 + 2^2 + 3^2
\end{aligned}$$

Therefore, the desired positive integer n is 1, 2, 3, 6, 7, or 15.

Problem 20. *Given that the product of the digits of a natural number x is equal to $44x - 86868$, and the sum of its digits is a perfect cube. Then the sum of all such natural numbers x is* __________.

Answer: 1989
Reasoning: Since $44x \geq 86868$, we have $x \geq \left[\frac{86868+43}{44}\right] = 1975$.

Thus, x is at least a four-digit number.

On the other hand, if x has $k \geq 5$ digits, then $44x - 86868 > 4 \times 10^k - 10^5 \geq 3 \times 10^k > 9^k$, which implies $44x - 86868 > p(x)$, where $p(x)$ is the product of the k digits of x. This is a contradiction, so x is exactly a four-digit number.

Given that the sum of the digits $S(x)$ satisfies $1 \leq S(x) \leq 36$, we have $S(x) = 1, 8$, or 27. Obviously, $S(x) = 1$ is not valid.

Since $0 < p(x) \leq 9^4 = 6561$, we have $x \leq \left[\frac{86868+6561}{44}\right] = 2123$.

The only possibilities for x satisfying $1975 \leq x \leq 2123$, $S(x) = 8$ or 27, and $p(x) \neq 0$ are 1989, 1998, 2114, and 2123. After checking, we find that only $x = 1989$ satisfies the given condition, where the product of its digits equals $44x - 86868$. Therefore, $x = 1989$ is the unique solution to this problem.

Hence, the sum of all such natural numbers x is 1989.

Problem 21. *Given a is a prime number and b is a positive integer such that $9(2a+b)^2 = 509(4a+511b)$, we need to find the values of a and b.*

Answer: 251,7
Reasoning:

Since $9(2a+b)^2 = 3^2(2a+b)^2$ is a perfect square, it follows that $509(4a+511b)$ must also be a perfect square. Since 509 is a prime number, we can express $4a+511b$ as $509 \times 3^2k^2$.

Hence, the original equation becomes $9(2a+b)^2 = 509^2 \times 3^2k^2$, which simplifies to $2a+b = 509k$.

Substituting $b = 509k - 2a$ into the equation , we get $4a + 511(509k - 2a) = 509 \times 3^2k^2$. Solving this equation yields $a = \frac{k(511-9k)}{2}$.

Since a is prime, $\frac{k(511-9k)}{2}$ must also be prime. Thus, we consider the following cases:

(1) When $k = 1$, $a = \frac{k(511-9k)}{2} = \frac{511-9}{2} = 251$ is prime, and $b = 509k - 2a = 509 - 2a = 7$.

(2) When $k = 2$, $a = \frac{k(511-9k)}{2} = \frac{511-18}{2} = 493 = 17 \times 29$, which is not prime.

(3) When $k > 2$ and k is odd, $a = \frac{k(511-9k)}{2} = k \cdot \frac{511-9k}{2}$ is prime. Since $k > 1$, $\frac{511-9k}{2} = 1$, but this equation has no integer solutions.

(4) When $k > 2$ and k is even, $a = \frac{k(511-9k)}{2} = \frac{k}{2}(511 - 9k)$ is prime. Since $\frac{k}{2} > 1$, $511 - 9k = 1$, but this equation has no integer solutions.

Thus, we conclude that $a = 251$ and $b = 7$.

Problem 22. *Let $\varphi(n)$ denote the number of natural numbers coprime to and less than n. Then, when $\varphi(pq) = 3p + q$, what is the sum of p and q?*

Answer: 14
Reasoning: (1) Proof: When p and q are distinct primes, $\varphi(pq) = (p-1)(q-1)$.

Since p and q are distinct primes, the natural numbers less than pq and coprime to pq cannot be divisible by p or q.

Out of the $pq-1$ natural numbers from 1 to $pq-1$, $q-1$ numbers are divisible by p and $p-1$ numbers are divisible by q. Thus,

$$\varphi(pq) = pq - 1 - (p-1) - (q-1) = (p-1)(q-1).$$

(2) From (1), we have $(p-1)(q-1) = 3p + q$, which simplifies to $pq - 4p - 2q + 1 = 0$, or $(p-2)(q-4) = 7$.

$$\begin{cases} p-2 = 1, \\ q-4 = 7, \end{cases}$$

$$\begin{cases} p-2 = 7, \\ q-4 = 1, \end{cases}$$

$$\begin{cases} p-2 = -1, \\ q-4 = -7, \end{cases}$$

$$\begin{cases} p-2 = -7, \\ q-4 = -1. \end{cases}$$

Solving these equations, we get $p = 3$, $q = 11$.

Hence, the sum of p and q is 14.

Problem 23. *Consider the sequence $\{S_n\}$ constructed as follows: $S_1 = \{1, 1\}$, $S_2 = \{1, 2, 1\}$, $S_3 = \{1, 3, 2, 3, 1\}$, and in general, if $S_k = \{a_1, a_2, \cdots, a_n\}$, then $S_{k+1} = \{a_1, a_1 + a_2, a_2, a_2 + a_3, \cdots, a_{n-1} + a_n, a_n\}$. What is the number of terms equal to 1988 in S_{1988}?*

Answer: 840
Reasoning: Let $\varphi(n)$ denote the Euler's totient function, which counts the number of positive integers less than n that are coprime to n.

It's easy to observe that in S_n, every pair of adjacent numbers are coprime, and the larger number equals the sum of its two adjacent neighbors.

Now, we use mathematical induction to prove that for $n \geq 2$, each pair of coprime numbers a and b less than or equal to n appears adjacent exactly twice in $S_2, S_3, \cdots, S_n$.

By symmetry, it suffices to show that a and b appear adjacent exactly once to the left of n. The case $n = 2$ is obvious.

Assume the result holds for $n - 1$ ($n - 1 \geq 2$). Consider only the left of 2.

If $a < n$, then a and b are not adjacent in S_n.

Otherwise, if a and b are adjacent in S_n, then $\alpha - b$ and b are both in S_{n-1} and adjacent. By the inductive hypothesis, α and b are adjacent in $S_2, S_3, \cdots, S_{n-1}$. Thus, $\alpha - b$ and b are adjacent in $S_2, S_3, \cdots, S_{n-2}$. This implies that $a - b$ and b are adjacent in $S_2, S_3, \cdots, S_{n-1}$ at least twice, a contradiction.

Therefore, a and b appear adjacent exactly once to the left of n. By symmetry, they appear adjacent exactly once to the right of n as well. Hence, a and b appear adjacent exactly twice in $S_2, S_3, \cdots, S_n$.

Now, consider those first occurrences of n in $S_2, S_3, \cdots, S_n$ (not limited to the left of 2). These are the numbers in S_n coprime to n, less than n, and their sum equals n. By the above argument, there are exactly $2\varphi(n)$ numbers adjacent to n. Thus, the number of occurrences of n in S_n is $\varphi(n)$.

Hence, in S_{1938}, the number of occurrences of 1988 is $\varphi(1988) = \varphi(4)\varphi(7)\varphi(71) = 840$.

Problem 24. *For a natural number n, let $S(n)$ denote the sum of its digits. For example, $S(611) = 6 + 1 + 1 = 8$. Let a, b, and c be three-digit numbers such that $a + b + c = 2005$, and let M be the maximum value of $S(a) + S(b) + S(c)$. How many sets (a, b, c) satisfy $S(a) + S(b) + S(c) = M$?*

Answer: 17160
Reasoning: Let $a = 100a_3 + 10a_2 + a_1$, $b = 100b_3 + 10b_2 + b_1$, and $c = 100c_3 + 10c_2 + c_1$, where $1 \leq a_3, b_3, c_3 \leq 9$ and $0 \leq a_2, b_2, c_2, a_1, b_1, c_1 \leq 9$.

Define $i = a_1 + b_1 + c_1$, $j = a_2 + b_2 + c_2$, and $k = a_3 + b_3 + c_3$. Given the conditions of the problem, we have $i + 10j + 100k = 2005$, and i, j, k are each less than or equal to 27.

We find the possible values of (i, j, k): $(i, j, k) = (5, 0, 20), (5, 10, 19), (5, 20, 18), (15, 9, 19), (15, 19, 18), (25, 8, 19), (25, 18, 18)$.

Thus, when $(i, j, k) = (25, 18, 18)$, $S(a) + S(b) + S(c) = i + j + k$ is maximized.

For $i = 25$, the possible pairs (a_1, b_1, c_1) are $(7, 9, 9)$, $(8, 8, 9)$, and their permutations, giving $3 \times 2 = 6$ possible pairs.

For $j = 18$, the possible pairs (a_2, b_2, c_2) are:

$$(0, 9, 9), (1, 8, 9), (2, 7, 9), (2, 8, 8), (3, 6, 9), (3, 7, 8),$$

$$(4,5,9),(4,6,8),(4,7,7),(5,5,8),(5,6,7),(6,6,6)$$

and their permutations,

so there are $6\times7+3\times4+1=55$ possible pairs.

For $k=18$, the possible pairs (a_3,b_3,c_3) are the same as (a_2,b_2,c_2), except $(0,9,9)$ and its permutations are excluded. Therefore, there are $55-3=52$ possible pairs for (a_3,b_3,c_3).

Hence, the number of sets (a,b,c) satisfying the condition is $6\times55\times52=17160$.

Problem 25. *For a natural number n, let $K(n,0)=\varnothing$. For any non-negative integers m and n, define $K(n,m+1)$ as the set of elements k such that $1\le k\le n$ and $K(k,m)\cap K(n-k,m)=\varnothing$, then the set $K(2004,2004)$ contains* _________ *elements.*

Answer: 127

Reasoning: Let's first list out some terms and try to identify a pattern:

$K(1,m)=1$,$K(2,m)=2$;

$K(3,m)=\{1,2,3\}$, $K(4,m)=\{4\}$;

$K(5,m)=\{1,4,5\}$, $K(6,m)=\{2,4,6\}$;

$K(7,m)=\{1,2,3,4,5,6,7\}$, $K(8,m)=\{8\}$;

$K(9,m)=\{1,8,9\}$, $K(10,m)=\{2,8,10\}$;

$K(11,m)=\{1,2,3,8,9,10,11\}$;

$K(12,m)=\{4,8,12\}$;

$K(13,m)=\{1,4,5,8,9,12,13\}$;

$K(14,m)=\{2,4,6,8,10,12,14\}$;

$K(15,m)=\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$,$K(16,m)=\{16\}$.

It seems $K(n,m)$ only depends on n, not on m.

Now, let's prove two lemmas using mathematical induction.

Lemma 1: $K(2n,m)=\{2j|j\in K(n,m)\}$.

Proof: Assume the proposition holds for positive integers less than n.

For $i=1$, it's clear that $1\in K(2n-1,m)$ and $1\notin K(2n,m)$, then $n\in\mathbb{N}_+$ implies $2j+1\notin K(2n,m)$.

Also, $2j\in K(2n,m)\iff K(2j,m)\cap K(2n-2j,m)=\varnothing$

$\iff K(j,m)\cap K(n-j,m)=\varnothing$ (by induction hypothesis)

$\iff j\in K(n,m)$.

Lemma 2: $K(2^n+i,m)=\{2^n+i\}\cup K(i,m)\cup\{2^n+i-j|j\in K(i,m)\}$, where $1\le i<2^n$.

Proof: Assume the proposition holds for positive integers less than 2^n+i.

For any $j\in K(i,m)$, if $j<i$, then

$K(j,m)\cap K(i-j,m)=\varnothing$

$\Rightarrow K(j,m)\cap K(2^n-i,m)=\varnothing$ (by induction hypothesis)

$\Rightarrow j\in K(2^n+i,m)$.

If $j=i$, then $i\in K(i,m)$ and $K(j,m)\cap\{2^n\}=\varnothing$, so

$K(i,m)\cap K(2^n,m)=\varnothing$ and thus $i\in K(2^n+i,m)$.

So, $K(i,m)\subset K(2^n+i,m)$.

For any $j\in K(2^n+i,m)$, if $j<\frac{2^n+i}{2}$, then

$K(j,m)\cap K(2^n+i-j,m)=\varnothing$

$\Rightarrow j\in K(i,m)$ (by induction hypothesis).

If $j=i+k$ and $k<2^{n-1}$, then we need to prove $K(i+k,m)\cap K(2^n-k,m)\neq\varnothing$.

In binary representation, if $j < \frac{2^n+i}{2}$ and there is no carry when adding i and k, then $2^a \in K(i+k,m) \cap K(2^n-k,m)$;

otherwise, if there is a carry at position $t < n$, then $2^t \in K(i+k,m) \cap K(2^n-k,m)$. This concludes the proof of Lemma 2.

Now, let's answer the original question. From Lemma 1 and Lemma 2, we have:

$$\begin{aligned}|K(2004,m)| &= |K(1002,m)| = |K(501,m)| = |K(256+245,m)| \\ &= 2|K(245,m)|+1 = 2|K(128+117,m)|+1 \\ &= 4|K(117,m)|+3 = 4|K(64+53,m)|+3 \\ &= 8|K(53,m)|+7 = 8|K(32+21,m)|+7 \\ &= 16|K(21,m)|+15 = 16|K(16+5,m)|+15 \\ &= 32|K(5,m)|+31 = 32\times 3+31 = 127.\end{aligned}$$

So, there are 127 elements in the set $K(2004,2004)$.

Problem 26. *Let $T = \{0,1,2,3,4,5,6\}$ and $M = \left\{\frac{a_1}{7}+\frac{a_2}{7^2}+\frac{a_3}{7^3}+\frac{a_4}{7^4}\middle| a_i \in T, i=1,2,3,4\right\}$. If the elements of M are arranged in descending order, then the 2005th number is __________.*

A) $\frac{5}{7}+\frac{5}{7^2}+\frac{6}{7^3}+\frac{3}{7^4}$ *B)* $\frac{5}{7}+\frac{5}{7^2}+\frac{6}{7^3}+\frac{2}{7^4}$

C) $\frac{1}{7}+\frac{1}{7^2}+\frac{0}{7^3}+\frac{4}{7^4}$ *D)* $\frac{1}{7}+\frac{1}{7^2}+\frac{0}{7^3}+\frac{3}{7^4}$

Answer: C

Reasoning: Let $[a_1a_2\cdots a_k]_p$ denote a k-digit number in base p. Multiplying each number in set M by 7^4, we get:

$M' = \left\{a_1\cdot 7^3 + a_2\cdot 7^2 + a_3\cdot 7 + a_4 \mid a_i \in T, i=1,2,3,4\right\} = \{[a_1a_2a_3a_4], |a_i \in T, i=1,2,3,4\}$.

The largest number in M' is $[6666]_7 = [2400]_{10}$.

In decimal, the 2005th number in descending order from 2400 is 2400 - 2004 = 396. And $[396]_{10} = [1104]_7$, dividing this number by 7^4, we obtain the numbers in M as $\frac{1}{7}+\frac{1}{7^2}+\frac{0}{7^3}+\frac{4}{7^4}$. Thus, option C is selected.

Problem 27. *Mutually prime positive integers p_n, q_n satisfy $\frac{P_n}{q_n} = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$. The sum of all positive integers n such that $3|p_n$ is __________.*

Answer: 31

Reasoning: Express n in ternary representation:

$n = \left(a_ka_{k-1}\cdots a_0\right)_3 = a_k\cdot 3^k + \cdots + a_1\cdot 3^1 + a_0$,

where $a_j \in \{0,1,2\}, j=0,1,2,\cdots,k, \alpha_k \neq 0$.

Let A_n denote the least common multiple of $1,2,\cdots,n$, then $A_n = 3^k\cdot B_n, 3\backslash B_n$.

Let $L_n = A_n\cdot\frac{P_n}{q_n} = A_n\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$, then $L_n \in \mathbf{N}_+$, and $3|p_n \Leftrightarrow 3^{b+1}|L_n$.

Let $S_j = \sum_{1\le i\le \frac{n}{3^j}}\frac{1}{i}, j=0,1,2,\cdots,k$, then

$$L_n = 3^k\cdot B_n\sum_{1\le i\le n}\frac{1}{i} = B_n\cdot S_k + 3^1\cdot B_n\cdot S_{k-1} + \cdots + 3^k\cdot B_n\cdot S_{\text{o}}.(*)$$

Lemma: When $a_i = 0$ or 2, $B_i \cdot S_j \equiv 0 (\text{mod} 3)$; when $a_i = 1$, $B_n \cdot S_j \equiv B_n \,(\text{mod}3)$.
Proof: Since $\frac{1}{3m+1} + \frac{1}{3m+2} = \frac{3(2m+1)}{(3m+1)(3m+2)}$, we have $B_n \cdot \left(\frac{1}{3m+1} + \frac{1}{3m+2}\right) \equiv 0(\text{mod}3)$
So when $a_j = 0$ or 2, $B_n \cdot Sj \equiv 0(\text{mod}3)$; when $a_j = 1$, $B_n S_j \equiv \frac{B_n}{3r+1} \equiv B_n \,(\text{mod}3)$.
Returning to the original question, suppose $3^{k+1} | L_n$. From (*), we have $B_n S_k \equiv 0(\text{mod}3)$.
From the lemma, we know $a_k = 2, S_k = \frac{3}{2}$. If $k = 0$, then $n = 2$.
When $k \geq 1$, from (*), we have

$$0 \equiv B_n \cdot \frac{3}{2} + 3^1 \cdot B_n \cdot S_{k-1} \big(\text{mod}9\big), \text{ so } 0 \equiv B_n \cdot S_{k-1} + B_n \cdot \frac{1}{2} \equiv B_n \cdot S_{k-1} - B_n \,(\text{mod}3)\,,$$

thus $B_n \cdot S_{k-1} \equiv B_n \,(\text{mod}3)$. From the lemma, we know $a_{k-1} = 1, S_{k-1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{7}$.
If $k = 1$, then $n = (2, 1)_3 = 7$.
When $k \geq 2$, from (*), we have

$$\begin{aligned}
&0 \equiv B_n \cdot \frac{3}{2} + 3^1 \cdot B_n \cdot S_{k-1} + 3^2 \cdot B_n \cdot S_{k-2} \,(\text{mod}27)\ , \\
&\text{so } 0 \equiv 3 \cdot B_n \cdot S_{k-2} + B_n \cdot \frac{1}{2} + B_n \cdot \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{7}\right) \\
&\equiv 3 \cdot B_n \cdot S_{-2} + B_n \cdot \left(2 + \frac{1}{4} + \frac{1}{5} + \frac{1}{7}\right) \\
&\equiv 3 \cdot B_n \cdot S_{k-2} + B_n\,(2 - 2 + 2 + 4) \equiv 3 \cdot (B_n \cdot S_{k-2} - B_n)\,(\text{mod}9)\,.
\end{aligned}$$

So $B_n \cdot S_{k-2} \equiv B_n \,(\text{mod}3)$. From the lemma, we know $a_{i-2} = 1, S_{i-2} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{22}$.
If $k \geq 3$, from (*), we have

$$\begin{aligned}
&0 \equiv B_n \cdot \frac{3}{2} + 3^1 \cdot B_n \cdot \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{7}\right) + 3^2 \cdot B_n \cdot S_{k-2} + 3^3 \cdot B_n \cdot S_{k-3}\,(\text{mod}81)\,, \\
&\text{so } 0 \equiv B_n \left(2 + \frac{1}{4} + \frac{1}{5} + \frac{1}{7}\right) + 3 \cdot B_n \cdot S_{k-2} + 3^2 \cdot B_n \cdot S_{k-3} \equiv B_n\,(2 + 7 + 11 + 4) \\
&+ 3 \cdot B_n \cdot S_{k-2} + 3^2 \cdot B_n \cdot S_{k-3} \\
&\equiv -3 \cdot B_n + 3 \cdot B_n \cdot S_{k-2} + 3^2 \cdot B_n \cdot S_{k-3}\,(\text{mod}27)\,,
\end{aligned}$$

$$\begin{aligned}
&\text{from which } 0 \equiv 3 \cdot B_m \cdot S_{k-3} + B_n \left(-1 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{22}\right) \\
&\equiv 3 \cdot B_n \cdot S_{k-3} + B_n \left[-1 + \left(1 + \frac{1}{2} + \frac{1}{4} - \frac{1}{4} - \frac{1}{2} - 1\right) \times 2 + \left(1 + \frac{1}{2} + \frac{1}{4}\right)\right] \\
&\equiv 3 \cdot B_n \cdot S_{k-3} + B_n\,(5 - 2) \equiv 3 \cdot B_n \cdot S_{k-3} + 3 \cdot B_n\,(\text{mod}9)\,.
\end{aligned}$$

Thus $B_n \cdot S_{k-3} + B_n \equiv 0(\text{mod}3)$, and from the lemma, we know this is impossible. Therefore, the sought-after positive integers n are 2, 7, and 22.

Problem 28. *Given x and y are prime numbers. The sum of the values of y in the solutions of the indeterminate equation $x^2 - y^2 = xy^2 - 19$ is* __________.

Answer: 10
Reasoning: If $x = y$, then there are obviously no solutions to the equation.

From the given equation, we have $x^y \equiv -19(\text{mod}y)$. Since x and y are both coefficients and $x \neq y$, then $(x, y) = 1$. By Fermat's Little Theorem, we have $x^{y-1} \equiv 1(\text{mod}y)$. Thus, we have $x + 19 \equiv 0\,(\text{mod}y)$.

Similarly, $19 - y \equiv 0\,(\text{mod}x)$. As $x - y + 19 \equiv 0\,(\text{mod}y)$ and $x - y + 19 \equiv 0\,(\text{mod}x)$, we get $x - y + 19 \equiv 0(\text{mod}xy)$.

It is evident that $x - y + 19 \neq 0$, thus $x + y + 19 > |x - y + 19| \geq xy$, implying $(x-1)(y-1) < 20$. Therefore, $|x - y| < 19$ and $x - y + 19 \geq xy$, i.e., $(x + 1)(y - 1) \leq 18$.

So, when $x \geq 5$, we have either $y = 2$ or $y = 3$. However, $x^2 - 2^x < 0$, $x^3 - 3^x < 0$, and $xy^2 - 19 > 0$, leading to contradictions. Hence, $x \leq 4$.

It can be verified that the solutions to the original indeterminate equation are (2,3) and (2,7).

Problem 29. *The number of non-zero integer pairs (a, b) for which $(a^3 + b)(a + b^3) = (a + b)^4$ holds is _________.*

Answer: 6
Reasoning: Note that $(a^3 + b)(a + b^3) = (a + b)^4$

$$\begin{aligned}
&\Leftrightarrow a^4 + a^3b^3 + ab + b^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
&\Leftrightarrow a^3b^3 + 2a^2b^2 + ab = 4a^3b + 8a^2b^2 + 4ab^3 \\
&\Leftrightarrow ab(ab + 1)^2 = 4ab(a + b)^2 \\
&\Leftrightarrow ab\left[(ab + 1)^2 - 4(a + b)^2\right] = 0
\end{aligned}$$

Hence, $(a, 0)$ and $(0, b)$ are solutions to the given equation, where $a, b \in \mathbf{Z}$. Additional solutions must satisfy $(ab + 1)^2 - 4(a + b)^2 = 0$. Since $(ab + 1)^2 - 4(a + b)^2 = 0$, either $ab + 1 = 2(a + b)$ or $ab + 1 = -2(a + b)$. We consider two cases:

1. If $ab + 1 = 2(a + b)$, then we have $(a - 2)(b - 2) = 3$. Thus, we have

$$\begin{cases} a - 2 = 3, \\ b - 2 = 1 \end{cases} \text{ or } \begin{cases} a - 2 = 1, \\ b - 2 = 3 \end{cases} \text{ or } \begin{cases} a - 2 = -3, \\ b - 2 = -1 \end{cases} \text{ or } \begin{cases} a - 2 = -1, \\ b - 2 = -3. \end{cases}$$

Solving these, we get

$a = 5, b = 3; a = 3, b = 5; a = -1, b = 1; a = 1, b = -1.$

If $ab + 1 = -2(a + b)$, then we have $(a + 2)(b + 2) = 3$.

Similarly, solving, we get

$a = 1, b = -1; a = -1, b = 1; a = -5, b = -3; a = -3, b = -5.$

In summary, the set of all possible solutions to the given equation is

$\{(a, 0) \mid a \in \mathbf{Z}\} \bigcup \{(0, b) \mid b \in \mathbf{Z}\} \cup \{(-5, -3), (-3, -5), (-1, 1), (1, -1), (3, 5), (5, 3)\}.$

Among them, there are 6 pairs of non-zero integer solutions: $(-5, -3), (-3, -5), (-1, 1), (1, -1), (3, 5), (5, 3)$.

Problem 30. *If the sum of the digits of a natural number α equals 7, then a is called an "auspicious number". Arrange all "auspicious numbers" in ascending order $a_1, a_2, a_3, \cdots$, if $a_n = 2005$, then a_{5n}=_________.*

Answer: 52000
Reasoning: Because the number of non-negative integer solutions of the equation $x_1+x_2+\cdots+x_t=m$ is C^m+t-1, and the number of integer solutions satisfying $x_1\geq 1$, $x_i\geq 0$ $(i\geq 2)$ is C_{m-2}^{m-1-2}. Now taking $m=7$, we can find that the number of k-digit "auspicious numbers" is $P(k)=C_{k+5}^{6}$.

2005 is the smallest "auspicious number" of the form $\overline{2abc}$, and $P(1)=C_6^6=1$, $P(2)=C_6^7=7$, $P(3)=C_6^8=28$. For the four-digit "auspicious numbers" $\overline{1abc}$, the number of such numbers satisfying $a+b+c=6$ is the number of non-negative integer solutions, which is $C_{6+3-1}^{6}=28$.

Because 2005 is the 1st+7th+28th+28th+1st=65th "auspicious number", i.e., $a_{65}=2005$, so $n=65$, $5n=325$.

$P(4)=C_9^6=84$, $P(5)=C_{10}^6=210$, and $\sum_{k=1}^{6}P(k)=330$.

So, the last six five-digit "auspicious numbers" from largest to smallest are: 70000,61000, 60100,60010,60001,52000.

Hence, the 325th "auspicious number" is 52000, i.e., $a_{5n}=52000$.

Problem 31. *The number of integers n in the interval $1\leq n\leq 10^6$ such that the equation $n=x^y$ has non-negative integer solutions x,y, and $x\neq n$ is* __________.

Answer: 1111
Reasoning: Let $N\left(x^y\right)$ represent the number of integers x^y.

If $1<x^5\leq 10^6$, using the principle of inclusion-exclusion, we have

$$N\left(x^y\right)=N\left(x^2\right)+N\left(x^3\right)+N\left(x^5\right)+N\left(x^7\right)+N\left(x^{11}\right)+N\left(x^{13}\right)+N\left(x^{17}\right)+N\left(x^{19}\right)-N\left(x^6\right)$$
$$-N\left(x^{10}\right)-N\left(x^{14}\right)-N\left(x^{15}\right)-N\left(x^{15}\right).$$

Since there are 10^3-1 square numbers greater than 1 and less than or equal to 10^6, we have $N\left(x^2\right)=999$. Similarly, there are 10^2-1 square numbers greater than 1 and less than or equal to 10^6, i.e., $N\left(x^3\right)=99$.

Since $15^5=819375<10^6$, there are $15-1$ fifth power numbers greater than 1 and less than or equal to 10^6, i.e., $N\left(x^5\right)=14$. Continuing this pattern, we can deduce that when $1<x^y\leq 10^6$,

$$N\left(x^y\right)=999+99+14+6+2+1+1-9-2-1-1=1110.$$

Additionally, when $n=1$, there is a non-negative integer solution with $x>1$ and $y=0$. Therefore, the number of integers n satisfying the conditions is 1111.

Problem 32. *Let p be a real number. If all three roots of the cubic equation $5x^3-5(p+1)x^2+(71p-1)x+1=66p$ are natural numbers, then the sum of all possible values of p is* __________.

Answer: 76
Reasoning:
Approach 1: Since $5-5(p+1)+(77p-1)+1=66p$, $x=1$ is a natural number solution of the original cubic equation.

By synthetic division, the cubic equation is reduced to the quadratic equation $5x^2-5px+66p-1=0(1)$.

This problem is transformed into finding all real numbers p such that the equation (1) has two natural number solutions.

Let u and $v(u \leq v)$ be the two natural number solutions of equation (1). By Vieta's formulas, we have:

$$\begin{cases} v+u=p, (2) \\ vu=\frac{1}{5}(66p-1).(3) \end{cases}$$

From (2) and (3), we eliminate p to get $5uv=66(u+v)-1$.(4)

From the condition, neither u nor v can be divisible by 2, 3, or 11. So, $u \geq 14$.

Furthermore: Since 49, $\nu=\frac{66u-1}{5u-66}$.(5) $u>66/5$ and $u \geq 14$ imply $u \geq 17$.

Since 2, 3, and 11 do not divide u, $u \geq 17$.

Also from ⊑≥u, we get $\frac{66u-1}{5u-66} \geq u$.

Thus, $5u^2-132u+1 \leq 0$, and $u \leq \frac{66+\sqrt{66^2-5}}{5}<\frac{132}{5}$.

Since 2, 3, and 11 do not divide u, u can only be 17, 19, 23, or 25.

By solving (5) when $u=17,19,23,25$, we find that when $u=17$, $\nu=59$, and both are natural numbers.

Approach 2: From equation (5) in Approach 1, we get $v=\frac{66u-1}{5u-66}=13+\frac{u+857}{5u-66}=13+\frac{1}{5}\left(1+\frac{4351}{5u-66}\right)=13+\frac{1}{5}\left(1+\frac{19\times229}{5u-66}\right)$.

To ensure v is an integer, $5u-66$ must divide 19 or 229.

By considering the prime factors of 19 and 229, we find that $5u-66=19$ when $u=17$, and $5u-66=229$ when $u=59$. Since $5u-66=229$ leads to $v \notin \mathbb{N}$, we discard it.

Therefore, $p=u+v=76$.

Approach 3: From Approach 1, for equation (1) to have natural numbers as solutions, $\Delta=25p^2-4\times5(66p-1)$ must be a perfect square.

Assuming $25p^2-20(66p-1)=q^2$, we get $(5p-132)^2-17404=q^2$. Let $5p-132=m$, then $m^2-q^2=17404$.

This implies both m and q are even. Setting $m=2m_0$ and $q=2q_0$, we get $m_0^2-q_0^2=4351=19\times229$. Solving the system of equations derived from $m>q$, we find $m_0=\pm2176, \pm124$.

Thus, $5p-132=2m_0=\pm4352$ or 248. From equation (5) in Approach 1, p is a natural number. From (3), we conclude that both u and v are odd.

Since (2) implies p is even, we find $p=76$.

Problem 33. *The number of triples of positive integers (a,b,c) satisfying $a^2+b^2+c^2=2005$ and $a \leq b \leq c$ is* __________.

Answer: 7

Reasoning: Since any odd perfect square leaves a remainder of 1 when divided by 4, and any even perfect square is a multiple of 4, it follows that among three squares, there must be two even squares and one odd square.

Let $a=2m, b=2n, c=2k-1$, where m,n,k are positive integers. The original equation becomes:

$$m^2+n^2+k(k-1)=501(1)$$

Since the remainder when a square is divided by 3 can only be 0 or 1, two cases need to be considered.

(i) If $3|k(k-1)$, then both m and n must be multiples of 3. Let $m = 3m_1, n = 3n_1$, and $\frac{k(k-1)}{3}$ be an integer. This yields:

$$3m^2 + 3n^2 + \frac{k(k-1)}{3} = 167(2)$$

Solving for $\frac{k(k-1)}{3} \equiv 167 \equiv 2 \pmod 3$ gives $\frac{k(k-1)}{3} = 3r + 2$, and $k(k-1) = 9r + 6$. (3)Since $k \leq 22$, we can try $k = 3, 7, 12, 16, 21$, which lead to:

$$\begin{cases} k = 3, \\ m_1^2 + n_1^2 = 55 \end{cases}$$

$$\begin{cases} k = 7, \\ m_1^2 + n_1^2 = 51 \end{cases}$$

$$\begin{cases} k = 12, \\ m_1^2 + n_1^2 = 41 \end{cases}$$

$$\begin{cases} k = 16, \\ m_1^2 + n_1^2 = 29 \end{cases}$$

$$\begin{cases} k = 21, \\ m_1^2 + n_1^2 = 9 \end{cases}$$

Among these, only $k = 12$ and $k = 16$ have positive integer solutions.

For $k = 12$, we get $(m_1, n_1) = (4, 5)$, resulting in $a = 6m_1 = 24, b = 6n_1 = 30, c = 2k - 1 = 23$.

For $k = 16$, we get $(m_1, n_1) = (2, 5)$, resulting in $a = 6m_1 = 12, b = 6n_1 = 30, c = 2k - 1 = 31$.

(ii) If $3 \nmid k(k-1)$, then either k or $k-1$ must be divisible by 3. Hence, k is congruent to 2 modulo 3, and k can only be 2, 5, 8, 11, 14, 17, or 20.

For each of these values, we need to check if $\frac{k(k-1)}{3}$ yields an integer:

For $k = 2$, $m_1^2 + n_1^2 = 499$, which does not have a solution.

For $k = 5$, $m_1^2 + n^2 = 481$, which has solutions $(m, n) = (9, 20)$ or $(15, 16)$.

For $k = 8$, $m_1^2 + n_1^2 = 391$, which does not have a solution.

For $k = 11$, $m^2 + n^2 = 319$, which does not have a solution.

For $k = 14$, $m_1^2 + n_1^2 = 229$, which does not have a solution.

For $k = 17$, $m_1^2 + n_1^2 = 121 = 11^2$, resulting in $(m, n) = (2, 15)$.

For $k = 20$, $m^2 + n_1^2 = 121 = 11^2$, which does not have a solution.

Therefore, there are 7 solutions:

$(23, 24, 30), (12, 30, 31), (9, 18, 40), (9, 30, 32), (4, 15, 42), (15, 22, 36), (4, 30, 33)$. All of these satisfy the original equation.

Problem 34. *The maximum positive integer k that satisfies* $1991^k \mid 1990^{19911992} + 1992^{19911990}$ *is* ________.

Answer: 1991
Reasoning: First, let's prove by mathematical induction:

For any odd number $a \geq 3$, for all positive integers n, we have $(1+a)^{a^{\alpha}} = 1 + S_n a^{n+1}$, where S_n is an integer and $a \nmid S_n$.

For $n = 1$, we have $(1+a)^a = 1 + C_\mu^1 a + C_\mu^2 a^2 + \cdots + C_\mu^\mu a^a = 1 + a^2\left(1 + C_\mu^2 + C_\mu^3 a + \cdots + a^{a-2}\right)$. Since a is odd, $a | C_a^2$, thus $a | C_a^2 + C_a^3 \alpha + \cdots + \alpha^{\alpha-2}$, and therefore $a \nmid S_1 = 1 + C_a^2 + \cdots + a^{a-2}$. Hence, equation(1) holds for $n = 1$.

Assume that equation(1) holds for a natural number $n = k_0$. Then

$$(1+a)^{a_0+1} = \left[(1+a)^{k_0}\right]^a = \left(1 + S_{k_0} a^{k_0+1}\right)^a$$

$$= 1 + S_{k_0} a^{k_0+2} + C_u^2 S_{k_0}^2 a^{2k_0+2} + \cdots + S_{k_0} a^{a(k_0+1)} = 1 + S_{k_0+1} a^{k_0+2},$$

where $S_{k_0+1} = S_{k_0} + C_a^2 S_{k_0}^2 a^{k_0} + \cdots + S_{k_0}^i a^{a(k_0+1)-k_0-2}$. By the induction hypothesis, $a \nmid S_k$, hence $a \nmid S_{k_0+1}$. Therefore, equation(1) holds for $n = k_0 + 1$.

Hence, equation(1) holds for all natural numbers n. Similarly, we can prove:

For any odd number $b \geq 3$, for all positive integers n, we have $(b-1)^{b^n} = -1 + T_n b^{n+1}$, where T_n is an integer and $b \nmid T_n$.

Using(1)and (2), we can find integers S and T such that $1991^{\dagger} S$, $1991 \nmid T$, and

$$1990^{1991^{1992}} + 1992^{1991^{1990}} = T \cdot 1991^{1993} + S \cdot 1991^{1991} = 1991^{1991}\left(T \cdot 1991^2 + S\right).$$

Thus, the maximum k we seek is 1991.

Problem 35. *Let a and b be positive integers such that $79 \mid (a + 77b)$ and $77 \mid (a + 79b)$. Then the smallest possible value of the sum $a + b$ is _________.*

Answer: 193
Reasoning: Note that

$$79 \mid (a + 77b) \Leftrightarrow 79 \mid (a - 2b) \Leftrightarrow 79 \mid (39a - 78b) \Leftrightarrow 79 \mid (39a + b),$$

$$77 \mid (a + 79b) \Leftrightarrow 77 \mid (a + 2b) \Leftrightarrow 77 \mid (39a + 78b) \Leftrightarrow 77 \mid (39a + b),$$

so $79 \times 77 \mid (39a + b)$. Thus, $39a + b = 79 \times 77k$, where $k \in \mathbb{N}_+$.

Note that

$$39a + 39b = 79 \times 77k + 38b = (78^2 - 1)k + 38b = (78^2 - 39)k + 38(k + b).$$

So, $39 \mid (b + k)$, and we have $b + k \geq 39$. Therefore, $39\alpha + 39b \geq (78^2 - 39) + 38 \times 39$, which implies $a + b \geq 156 - 1 + 38 = 193$.

It is easy to see that $b = 38$ and $\alpha = 155$ satisfy the given conditions. Therefore, $a + b = 193$.

Hence, the minimum value of $(s + n)_{\min} = 193$.

Problem 36. *Let a_i, b_i $(i = 1, 2, \ldots, n)$ be rational numbers such that for any real number x, we have $x^2 + x + 4 = \sum_{i=1}^n (a_i x + b_i)^2$. Then the minimum possible value of n is _________.*

Answer: 5
Reasoning: Since $x^2 + x + 4 = \left(x + \frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + 1^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2$, it is clear that $n = 5$ is possible. We will now prove that $n = 4$ is not possible.

Proof by contradiction: Suppose $n = 4$. Let $x^2+x+4 = \sum_{i=1}^{4}(a_i x+b_i)^2$, where $a_i, b_i \in \mathbb{Q}$. Then,

$$\sum_{i=1}^{4} a_i^2 = 1, \quad \sum_{i=1}^{4} a_i b_i = \frac{1}{2}, \quad \text{and} \quad \sum_{i=1}^{4} b_i^2 = 4.$$

So,

$$\frac{15}{4} = (-\alpha_1 b_2 + \alpha_2 b_1 - \alpha_3 b_4 + \alpha_4 b_3)^2 + (-\alpha_1 b_4 + \alpha_3 b_1 - \alpha_1 b_2 + \alpha_2 b_1)^2 + (-\alpha_1 b_4 + \alpha_4 b_1 - \alpha_2 b_3 + \alpha_3 b_2)^2.$$

The above expression implies that $a^2 + b^2 + c^2 = 15d^2 \equiv -d^2 \pmod 8$ has a solution. Without loss of generality, assume that at least one of a, b, c, d is odd and $a^2, b^2, c^2, d^2 \equiv 0, 1, 4 \pmod 8$. It is clear that the above expression has no solution, leading to a contradiction. Hence, $n = 4$ is not possible.

Problem 37. *Let α and β be the two roots of the equation $x^2 - x - 1 = 0$. Define $\alpha_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ for $n = 1, 2, \ldots$. For some positive integers a and b, with $a < b$, if for any positive integer n, b divides $a_n - 2na^n$, then the sum of all such positive integers b is* __________.

Answer: 5
Reasoning:

1. First, let's prove that for any positive integer n, we have $\alpha_{n+2} = \alpha_{n+1} + \alpha_n$. Note that

   $$\alpha^{n+2} - \beta^{n+2} = (\alpha + \beta)(\alpha^{n+1} - \beta^{n+1}) - \alpha\beta(\alpha^n - \beta^n) = (\alpha^{n+1} - \beta^{n+1}) + (\alpha^n - \beta^n),$$

   which implies $a_{n+2} = a_{n+1} + a_n$.

2. Given the conditions, we know that b divides $\alpha_1 - 2\alpha$, i.e., $b|1 - 2\alpha$. Since $b > \alpha$, we have $b = 2\alpha - 1$. Furthermore, for any positive integer n, we have $b|\alpha_n - 2n\alpha^n$, $b|\alpha_{n+1} - 2(n+1)\alpha^{n+1}$, and $b|\alpha_{n+2} - 2(n+2)\alpha^{n+2}$. Combining these with $\alpha_{n+2} = \alpha_{n+1} + \alpha_n$ and $b = 2\alpha - 1$ being odd, we get

   $$b|(n+2)a^{n+2} - (n+1)a^{n+1} - na^n.$$

   Since $(b, a) = 1$, we have $b|(n+2)a^2 - (n+1)a - n$.

3. By setting n to $n+1$ in the above equation, we get $b|(n+3)a^2 - (n+2)a - (n+1)$. Subtracting this from the previous equation yields $b|a^2 - a - 1$, i.e., $2a - 1|a^2 - a - 1$. So, $2a - 1|2a^2 - 2a - 2$. Since $2\alpha^2 \equiv \alpha \pmod{2\alpha - 1}$, we have $2\alpha - 1| - \alpha - 2$ and $2\alpha - 1| - 2\alpha - 4$. Thus, $2\alpha - 1| - 5$, implying $2\alpha - 1 = 1$ or 5. However, $2\alpha - 1 = 1$ leads to $b = \alpha$, which is a contradiction. Hence, $2\alpha - 1 = 5$, giving $\alpha = 3$ and $b = 5$.

4. Now, we need to show that when $a = 3$ and $b = 5$, for any positive integer n, we have $5|(\alpha_n - 2n\alpha^n)$. For $n = 1, 2$, we have $\alpha_1 = 1$ and $\alpha_2 = \alpha + \beta = 1$, so $\alpha_1 - 2 \times 3 = -5$ and $\alpha_2 - 2 \times 2 \times 3^2 = -35$, which confirms the condition. Assuming the condition holds for $n = k, k+1$, we can prove it for $n = k+2$ as well. Hence, $(a, b) = (3, 5)$ satisfies the conditions.

Problem 38. *Let n be a natural number greater than 3 such that $1 + C_n^1 + C_n^2 + C_n^3$ divides 2^{2000}. Then, the sum of all such n satisfying this condition is* ________.

Answer: 30
Reasoning: Since 2 is a prime number, the problem is equivalent to finding natural numbers $n > 3$ such that

$$1 + C_n^1 + C_n^2 + C_n^3 = 2^k \text{ for some } k \in \mathbb{N}, k \le 2000.$$

We have

$$1 + C_n^1 + C_n^2 + C_n^3 = 1 + n + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{6} = \frac{(n+1)(n^2-n+6)}{6},$$

which means

$$(n+1)(n^2 - n + 6) = 3 \times 2^{k+1}.$$

Let's substitute $m = n + 1$, then we have

$$m(m^2 - 3m + 8) = 3 \times 2^{k+1}.$$

Now, let's consider different cases for m.

1. If $m = 2^s$ where $m > 4$ and $s \ge 3$, then $m^2 - 3m + 8 = 3 \times 2^t$ for some $t \in \mathbb{N}$. If $s \ge 4$, then $m^2 - 3m + 8 = 3 \times 2^t \equiv 8 \pmod{16}$. So, $t = 3$, which implies $m^2 - 3m + 8 = 24$, i.e., $m(m-3) = 16$, which is not possible. Thus, we have only $s = 3$, which gives $m = 8$, i.e., $n = 7$.

2. If $m = 3 \times 2^u$ where $m > 4$ and $u \ge 1$, then $m^2 - 3m + 8 = 2^\nu$ for some $\nu \in \mathbb{N}$. If $u \ge 4$, then $m^2 - 3m + 8 = 2^\nu \equiv 8 \pmod{16}$. So, $\nu = 3$, which implies $m(m-3) = 0$, which is not possible. Also, for $u = 1$ and $u = 2$, $m^2 - 3m + 8$ cannot be a power of 2. When $m = 3 \times 2^3 = 24$, we find $n = 23$.

Thus, the solutions are $n = 7$ and $n = 23$. Therefore, the sum is $7 + 23 = 30$.

Problem 39. *In the decimal representation, the product of the digits of k equals $\frac{25}{8}k - 211$. Then the sum of all positive integers k satisfying this condition is* ________.

Answer:160
Reasoning: Let k be a decimal number, and let s be the product of the digits of k.

It's easy to see that $s \in \mathbb{N}$, so 8 divides k and $\frac{25}{8}k - 211 \ge 0$, implying $k \ge \frac{1688}{25}$. Since $k \in \mathbb{N}_+$, we have $k \ge 68$.

Also, since 8 divides k, the units digit of k must be even, making s even as well. Since 211 is odd, $\frac{25}{8}k$ is odd, implying 16 divides k. Let $k = \overline{a_1 a_2 \cdots a_i}$, where $0 \le a_i \le 9$ for $i = 2, 3, \cdots, t$ and $1 \le a_1 \le 9$. By definition, we have:

$$S=\prod_{i=1}^{t} a_i \le a_1\times 9^{-i} < a_1\times 10^{-1} = \overline{a_1 \underbrace{00\cdots 0}_{t-1 \text{ digits}}} \le k\,.$$

Therefore, $k > s = \frac{25}{8}k - 211$, implying $k \le 99$.

Since 8 divides k and 16 does not divide k, we have $k = 72$ or 88. Upon verification, both $k = 72$ and $k = 88$ satisfy the given condition. Hence, the answer is $72 + 88 = 160$.

Problem 40. *Let n be an integer, and let $p(n)$ denote the product of its digits (in decimal representation). Then the sum of all n such that $10\,p(n) = n^2 + 4n - 2005$ is* __________.

Answer: 45
Reasoning: (1) First, we prove that $p(n) \le n$.

Assume n has $k+1$ digits, where $k \in \mathbb{N}$. Then $n = 10^k\alpha_k + 10^{k-1}\alpha_{k-1} + \cdots + 10\alpha_1 + \alpha_0$, where $a_1, a_2, \cdots, a_k \in \{1, 2, \cdots, 9\}$. Thus, we have $p(n) = a_9a_1\cdots a_k \le a_k 9^k \le a_k 10^k \le n$. Therefore, $p(n) \le n$.

(2) First, note that $n^2 + 4n - 2005 \ge 0$ implies $n \ge 43$. Furthermore, $n^2 + 4n - 2005 = 10\,p(n) \le 10n$ implies $n \le 47$. Hence, we deduce that $n \in \{43, 44, 45, 46, 47\}$. Upon checking each case, we find $n = 45$.

Problem 41. *There are some positive integers with more than two digits, such that each pair of adjacent digits forms a perfect square. Then the sum of all positive integers satisfying the above conditions is* __________.

Answer:97104
Reasoning: It is easy to observe that the perfect squares with two digits are: 16, 25, 36, 49, 64, 81.

Note that, starting from the given digits, there can be at most 1 two-digit perfect square. Therefore, after the first two-digit number is selected, the remaining part of the number is uniquely determined. Since there are no perfect squares starting with 5 or 9, the number cannot start with 25 or 81.

Starting from 16, we get 164 and 1649; From 36, we get 364 and 3649; From 64, we get 649; From 81, we get 816, 8164, and 81649.

Therefore, the numbers satisfying the condition are 164, 1649, 364, 3649, 649, 8164, and 81649.

Problem 42. *Let α be an integer, and $|\alpha| \le 2005$. The number of values of α that make the system of equations* $\begin{cases} x^2 = y + \alpha, \\ y^2 = x + \alpha \end{cases}$ *have integer solutions is* __________.

Answer: 90
Reasoning: If (x, y) is an integer solution to the given system of equations, subtracting the two equations gives

$$x^2 - y^2 = y - x \Longleftrightarrow (x-y)(x+y+1) = 0.$$

Consider the following two cases.

(1) When $x-y=0$. Let $x=y=m$ be substituted into the system of equations, resulting in $\alpha=m^2-m=m(m-1)$. It's easy to see that α is the product of two consecutive integers. Thus, α is non-negative, and these numbers do not exceed 2005. Moreover, $45\times 44=1980<2005$ and $46\times 45=2070>2005$. Since m can take all integers from 1 to 45, there are 45 values of α satisfying this condition.

(2) When $x+y+1=0$. Let $x=m$ and $y=-(m+1)$ be substituted into the system of equations, resulting in $a=m^2+m+1=m(m+1)+1$. It's easy to see that α is one greater than the product of two consecutive integers. Adding 1 to the α obtained in the first case gives the α in the second case. Again, there are 45 distinct values of α satisfying this condition.

In conclusion, there are a total of 90 values of α satisfying the condition.

Problem 43. *Divide the set $S=\{1,2,\cdots,2006\}$ into two disjoint subsets A and B such that:*

(1) $B\in A$;

(2) If $a\in A$ and $b\in B$ with $a+b\in S$, then $a+b\in B$;

(3) If $a\in A$, $b\in B$, and $ab\in S$, then $ab\in A$.

The number of elements in set A is _________.

Answer: 154

Reasoning: Clearly, $1\in B$ (if not, $1\in A$, and by condition (3), for any $b\in B$, $1\times b=b\in A$, contradiction). For any $a\in A$, by condition (2), $a+1\in B$, thus, for any $k\in\mathbb{N}$, $ka+1\in B$. Hence, $2\in B$ (if not, $2\in A$, and for any $k\in\mathbb{N}$, $2k+1\in B$, leading to $13\in B$, contradiction). Similarly, $3,4,6$, and $12\in B$, implying that any factor of $\alpha-1$ for $\alpha\in A$ belongs to B. By condition (3), for any $a\in A$, we have $2a,3a\in A$. Since $13\in A$, we have $13+1=14\in B$ (if not, $7\in B$, implying $14\in A$, contradiction). Also, $2\times 13+1=27\in B$, leading to $9\in B$. Similarly, $3\times 13+1=40\in B$, hence $20,10,5\in B$, and $8\in B$ (if not, $8\in A$, implying $8\times 5=40\in A$, contradiction). Moreover, $5\times 13+1=66\in B$, yielding $33,22,11\in B$. Thus, $\{1,2,\cdots,12\}\subseteq B$, and $13\in A$. By condition (2), for any $k\in\mathbb{N}$ and $i=1,2,\cdots,12$, we have $13k+i\in B$. By condition (3), for any $k\in\mathbb{N}$ and $i=1,2,\cdots,12$, we have $13(13k+i)\in A$, especially $13i\in A$ for $i=1,2,\cdots,12$. If $13^2t\in B$ for some $t\in\mathbb{N}_+$, then by condition (2), $13^2t+13i=13(13t+i)\in B$, contradiction. Therefore, for any $t\in\mathbb{N}_+$, $13^2t\in A$. Thus, $A=\left\{13t \,\middle|\, t=1,2,\cdots,\left\lfloor\frac{2006}{13}\right\rfloor\right\}$ and $B=S-A$. Upon inspection, these sets satisfy the conditions.

Problem 44. *Let S be a finite set of integers. Suppose that for any two distinct elements $p,q\in S$, there exist three elements a, b, $c\in S$ (not necessarily distinct, and $a\neq 0$) such that the polynomial $F(x)=\alpha x^2+bx+c$ satisfies $F(p)=F(q)=0$. The maximum number of elements in S is* _________

Answer: 3

Reasoning: It is easy to verify that $S=\{-1,0,1\}$ satisfies the condition. Now, we will prove that $|S|_{\max}=3$.

(1) At least one of 1 and -1 must belong to S. Conversely, assume $a_1,a_2\in S$ such that by the given condition, there exist $\alpha,b,c\in S$ satisfying $F(a_1)=F(a_2)=0$. Then, $\frac{c}{a}=a_1a_2\Rightarrow c=aa_1a_2$. Then, there exists $a_3=c\in S$, and repeating this process yields

$\alpha_i\ (i=1,2,\cdots)\in S$, but $|\alpha_1|\leq|\alpha_2|<|\alpha_3|<\cdots<|\alpha_k|<\cdots$, which contradicts the fact that S is a finite set.

(2) Without loss of generality, let $1\in S$. There exists $a_1\in S(\alpha\neq 1)$. Then, by the given condition, there exist $a,b,c\in S$ such that $a+b+c=0\Rightarrow b=-a-c$, and $a_1+1=-\frac{b}{a}=1+\frac{c}{a}$. Thus, $a_1=\frac{c}{a}\Rightarrow c=aa_1$.

(i) If $a_1\geq 2$, then for $\alpha\neq\pm1$, $|c|>|a_1|$. We can find $\alpha_2=c\in S(|a_2|>|a_1|)$, leading to $|a_1|<|a_2|<\cdots\in S$, which contradicts the finiteness of S. If $a=1,b=-a_1-1;a=-1,b=a_1+1$, then for any $\alpha=\pm1$, $|b|>|\alpha_i|$, which also leads to a contradiction.

(ii) If $a_1\leq-2$, consider $-\frac{b}{a}=a_1+1,\frac{c}{a}=a_1$. By the assumption, there is no $a\in S$ such that $a\geq 2$. Since $a_1\leq-2$, b and c have opposite signs. If $a\leq-2$, then $c>|a_1|\geq 2$, which is a contradiction. If $a=-1$, then $b=a_1+1,c=-a_1\geq 2$, which also contradicts. If $\alpha=1$, then $b=-a_1-1,c=a_1$. If $a_1\leq-3$, then $b\geq 2$, which is a contradiction. Thus, we conclude that $a_1\in\{-2,-1,0\}$.

It is evident that $S=\{-2,-1,0,1\}$ does not satisfy the condition. For example, for $-1,-2\in S$, $x^2+3x+2=0$, which is impossible. Therefore, $|S|_{\max}=3$.

Problem 45. *A natural number whose last four digits are 2022 and is divisible by 2003 has a minimum value of _____.*

Answer: 2672002
Reasoning: Set this number to be $10000x+2002$, then

$$\begin{aligned}x&=\frac{-2002}{10000}=\frac{1}{10000}=\frac{1+2003\times 3}{10000}\\&=\frac{601}{1000}=\frac{661}{100}=\frac{667}{10}\\&=267(\bmod 2003).\end{aligned}$$

So the sought value is 2672002.

Problem 46. *The number of positive integer solution in $\left(x^2+2\right)\left(y^2+3\right)\left(z^2+4\right)=60xyz$ is ______.*

Answer: 8
Reasoning: First, let's determine the upper bounds for x,y,z. Because,

$$\begin{aligned}\left(x^2+2\right)\left(y^2+3\right)&=x^2y^2+3x^2+2y^2+6\\&>\left(x^2y^2+4\right)+2\left(x^2+y^2\right)\\&\geqslant 4xy+4xy=8xy\end{aligned}$$

we have from the original equation

$$\begin{aligned}&8xy\left(z^2+4\right)<60xyz,\\&2z^2-15z+8<0.\end{aligned}$$

From (1), it is obvious that $z<8$, and since $z=7$ does not satisfy (1), we have $z\leqslant 6$. The right side of the original equation is divisible by 5, and since

$$\begin{aligned}&x^2\equiv 0,\pm1(\bmod 5),\\&x^2+2\equiv 1,2,3(\bmod 5),\\&y^2+3\equiv 2,3,4(\bmod 5),\end{aligned}$$

it must be that z^2+4 is divisible by 5. Thus $z \equiv \pm 1(\bmod 5)$, so$z=1,4,6$. If $z=6$, then

$$\left(x^2+2\right)\left(y^2+3\right)=9xy,$$

but

$$\left(x^2+2\right)\left(y^2+3\right) \geqslant 2\sqrt{2}x \cdot 2\sqrt{3}y=4\sqrt{6}xy>9xy,$$

contradiction.
If $z=4$, then $\left(x^2+2\right)\left(y^2+3\right)=12xy$. When x=1, $y^2+3=4y$, so y=1 or 3. When x=2, $y^2+3=4y$, so y=1 or 3. Thus $x \geqslant 3$, from (2)

$$12y=\left(x+\frac{2}{x}\right)\left(y^2+3\right)>x\left(y^2+3\right) \geqslant 3\left(y^2+3\right),$$

thus, $y^2-4y+3<0$, and hence, $y=2$. Then from (2) we obtain $7\left(x^2+2\right)=24x, 7 \mid x$, so $x \geqslant 7$. But $7\left(x^2+2\right)-24x>x(7x-24)>0$, which has no solution. If $z=1$, then we again arrive at (2). Thus, the total number of solutions for this problem is $2 \times 4=8$. The specific solutions are $(x,y,z)=(1,1,4),(1,3,4),(2,1,4),(2,3,4),(1,1,1),(1,3,1),(2,1,1),(2,3,1)$.

Problem 47. *The number of integers satisfying the condition that x^2+5n+1 is a perfect square is known to be ______.*

Answer: 4
Reasoning: When n is a positive integer, $(n+1)^2<n^2+5n+1<(n+3)^2$. Therefore

$$\begin{aligned}&n^2+5n+1=(n+2)^2=n^2+4n+4,\\&n=3.\end{aligned}$$

When n=0, $n^2+5n+1=1$ is a perfect square.
When n is a negative integer, let $m=-n$, then m is a positive interger, and $n^2+5n+1=m^2-5m+1$.
If $m \leqslant 4$, then $m^2-5m+1<0$. If $m=5$ (i.e., $n=-5$), then $m^2-5m+1=1$ is a perfect square.
If $m>5$, set $k=m-5$, then $m^2-5m+1=k(k+5)+1=k^2+5k+1$.
From the previous results, we know $k=3, m=8, n=-8$.
Therefore, the values of the integer n are $3,0,-5,-8$.

Problem 48. *If p, q, r are prime numbers such that $p+q+r=1000$, then the remainder when $p^2q^2r^2$ is divided by 48 is ______.*

Answer: 48
Reasoning: One of p,q,r must be 2. Without loss of generality, let's assume $r=2$. Then p and q are both not 2, and $p+q=1000-2$. Because 1000-2-3 is a multiple of 5 and not a prime number, both p and q are not 3.

$$\begin{aligned}&p^2 \equiv q^2 \equiv 1(\bmod 4),\\&p^2 \equiv q^2 \equiv 1(\bmod 3),\\&p^2q^2 \equiv 1(\bmod 12),\\&p^2q^2r^2 \equiv 4(\bmod 48),\end{aligned}$$

Problem 49. *Given x, y, z are integers, and $10x^3 + 20y^3 + 2006xyz = 2007z^3$, then the maximum of $x + y + z$ is _____.*

Answer: 0
Reasoning: The left side of the original equation consists of three even terms, so the right side $2007z^3$ is also even, implying z is even.

Since $2007z^3, 2006xyz$, and $20y^3$ are all divided by 4, $10x^3$ is also divided by 4, making x even.

Further, since $10x^3$,$2006xyz$, and $2007z^3$ are all divided by 8, $20y^3$ is also divisible by 8, making y even.

Dividing both sides of the original equation by 8, we get:

$$10\left(\frac{x}{2}\right)^3 + 20\left(\frac{y}{2}\right)^3 + 2006\left(\frac{x}{2}\right)\left(\frac{y}{2}\right)\left(\frac{z}{2}\right) = 2007\left(\frac{z}{2}\right)^3.$$

Similarly, $\frac{x}{2}, \frac{y}{2}, \frac{z}{2}$ are all even, Therefore, we can replace them with $\frac{x}{4}, \frac{y}{4}, \frac{z}{4}$. Continuing this process, we find that $\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n} \quad (where, n = 0, 1, 2, \cdots)$ are all integer solutions of the original equation. However, when $x \neq 0$, taking $2^n > |x|$ implies $\frac{x}{2^n}$ is not an integer, hence x=0 . Similarly, y=0, z=0 . Therefore, the only integer solution of the original equation is $x = y = z = 0$, leading to $x + y + z = 0$.

Problem 50. *For a positive integer n, if the first two digits of 5^n and 2^n are the same, denoted as a and b respectively, then the value of the two-digit number $\overline{ab}$ is _____.*

Answer: 31
Reasoning: Let this two-digit number be x, Then there exist positive integers k, h , such that

$$10^k \cdot x < 2^n < 10^k(x+1),$$
$$10^n \cdot x < 5^n < 10^h(x+1),$$

Multiplying both equations, we get $10^{k+h}x^2 < 10^n < 10^{k+h}(x+1)^2$. Since x is a two-digit number, $10^2 \leqslant x^2, \quad (x+1)^2 \leqslant 10^4$, so $10^{k+h+2} < 10^n < 10^{k+h+4}$, which implies n=k+h+3 . Canceling out 10^{k+h} , we get $x^2 < 10^3 < (x+1)^2$. Since $31^2 = 961, 32^2 = 1024$, we have x=31 , hence $\overline{ab} = 31$.

Problem 51. *The remainder when $\frac{2020\times2019\times\cdots\times1977}{44!}$ is divided by 2021 is _____.*

Answer: 1975
Reasoning: Firstly, the product of the consecutive 44 integers from 1977 to 2020 is divided by 44!, meaning $\frac{2020\times2019\times\cdots\times1977}{44!}$ is a positive integer. Secondly, 2021 is not a prime number, because $2021 = 43 \times 47$. Since 44! does not have a prime factor of 47 , and $42 \times 47 = 2021 - 47 = 1974$, thus

$$1977 = 3, 1978 = 4, \cdots, 2020 = 46,$$
$$\frac{2020\times2019\times\cdots\times1977}{44!} = \frac{46!\,\mathrm{div}\,2}{44!} = \frac{46\times45}{2}$$
$$\equiv \frac{(-1)\times(-2)}{2} \equiv 1 \equiv 1975(\mathrm{mod}47).$$

Also

$$\begin{aligned}\frac{2020\times 2019\times\cdots\times 1977}{44!} &= \frac{2020\times 2019\times\cdots\times 1979}{42!}\times\frac{1978\times 1977}{43\times 44}\\ &\equiv \frac{(-1)\times(-2)\times\cdots\times(-42)}{42!}\times\frac{46\times 1977}{44}\\ &\equiv \frac{3\times(-1)}{1} = -3\equiv 1975(\text{mod}43),\end{aligned}$$

Thus $\frac{2020\times 2019\times\cdots\times 1977}{44!}\equiv 1975(\text{mod}2021)$

Problem 52. *Given the sequence* $\{a_n\}: a_1=1, a_{n+1}=\frac{\sqrt{3}a_n+1}{\sqrt{3}-a_n}$*, then* $\sum\limits_{n=1}^{2022} a_n=$ _____.

Answer: 0
Reasoning: It is easy to obtain $a_1=1, a_2=2+\sqrt{3}, a_3=-2-\sqrt{3}, a_4=-1, a_5=-2+\sqrt{3}, a_6=-\sqrt{3}, a_7=1$. Then the sequence $\{a_n\}$ is periodic with a period of 6, therefore $\sum\limits_{n=1}^{2022} a_n=337(a_1+a_2+...+a_6)=0$.

Problem 53. *Given* $2bx^2+ax+1-b\geq 0$ *holds for* $x\in[-1,1]$*, find the maximum value of* $a+b$.

Answer: 2
Reasoning: From the problem statement, we know $xa+(2x^2-1)b\geq -1$ always holds for $x\in[-1,1]$. Taking $x=-\frac{1}{2}$, we get

$$-\frac{1}{2}(a+b)\geq -1\Rightarrow a+b\leq 2.$$

When $a=\frac{4}{3}$, $b=\frac{2}{3}$, $2bx^2+ax+1-b=\frac{3}{4}x^2+\frac{4}{3}x+\frac{1}{3}=\frac{1}{3}(2x+1)^2\geq 0$, which always holds for $x\in[-1,1]$. At this time, $a+b=2$. Therefore, the maximum value of $a+b$ is 2.

Problem 54. *Given* $x,y\in[0,+\infty)$*, and satisfying* $x^3+y^3+6xy=8$*. Then the minimum value of* $2x^2+y^2=$ _____.

Answer: $\frac{8}{3}$
Reasoning: According to Euler's formula $x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$, it is easy to know $x+y=2$, and from Cauchy's inequality, we know $2x^2+y^2\geq\frac{8}{3}$

Problem 55. *Given* $f(x)$ *and* $g(x)$ *are two quadratic functions with the coefficient of the quadratic term being 1 for both. If* $g(6)=35$, $\frac{f(-1)}{g(-1)}=\frac{f(1)}{g(1)}=\frac{21}{20}$*, then* $f(6)=$ _____.

Answer: 35
Reasoning: Let $f(x)=x^2+ax+b$, $g(x)=x^2+cx+d$. From the given condition we have:

$$\begin{aligned}20(1-a+b)=21(1-c+d),①\\ 20(1+a+b)=21(1+c+d),②\end{aligned}$$

From ① + ②, we have $40+40b=42+42d$, then $20b=1+21d$.// From ① − ②, we have $-40a=-42c$, then $20a=21c$. From $g(6)=35$, we have $36+6c+d=35$. So $36+6\times\frac{20}{21}a+\frac{20b-1}{21}=35$. Thus $6a+b=-1$. Then $f(6)=36+6a+b=35$.

Problem 56. *Given* $(n+1)^{a+1} - n^{a+1} < na^(a+1) < na^{n+1} - (n-1)^{a+1} \quad (-1 < a < 0)$①. *Let*$x = \sum_{k=4}^{106} \frac{1}{\sqrt[3]{k}}$, *then the integer part of* x *is* ___.

Answer: 14996
Reasoning: In ①, take $\alpha = -\frac{1}{3}, n = 4, 5, \cdots, 10^6$, by adding inequalities, we have $(10^6+1)^{\frac{2}{3}} - 4^{\frac{2}{3}} < \frac{2}{3}\sum_{k=4}^{10^5} \frac{1}{\sqrt[3]{k}} < (10^6)^{\frac{2}{3}} - 3^{\frac{2}{3}}$. Then the integer part of x is 14996.

Problem 57. *Let* $a_1 = \frac{\pi}{6}, a_n \in (0, \frac{\pi}{2})$, *and* $\tan a_{n+1} \cdot \cos a_n = 1 (n \geq 1)$. *If* $\prod_{k=1}^{m} \sin a_k = \frac{1}{100}$, *then* $m =$ ___.

Answer: 3333
Reasoning: From $\tan a_{n+1} \cdot \cos a_n = 1 \Rightarrow \tan^2 a_{n+1} - \tan^2 a_n = 1 \Rightarrow \tan^2 a_n - \tan^2 a_1 = n - 1 \Rightarrow \tan^2 a_n = n - 1 + \frac{1}{3} \Rightarrow sina_n = \frac{\sqrt{3n-2}}{\sqrt{3n+1}}$. From $\prod_{k=1}^{m} \sin a_k = \frac{1}{\sqrt{3m+1}} = \frac{1}{100}$, we have $m = 3333$.

Problem 58. *Let* $y = f(x)$ *be a strictly monotonically increasing function, and let its inverse function be* $y = g(x)$. *Let* x_1, x_2 *be the solutions to the equations* $f(x) + x = 2$ *and* $g(x) + x = 2$ *respectively. Then* $x_1 + x_2 =$ ___.

Answer: 2
Reasoning: Given that $f(x) + x$ is strictly monotonically increasing and $f(x_1) + x_1 = 2 = g(x_2) + x = f(g(x_2)) + g(x_2)$. Therefore, $x_1 = g(x_2), x_2 = f(x_1)$. Thus, $x_1 + x_2 = x_1 + f(x_1) = 2$.

Problem 59. *Let* $x_0 > 0, x_0 \neq \sqrt{3}, Q(x_0, 0), P(0, 4)$, *and the line* PQ *intersects the hyperbola* $x^2 - \frac{y^2}{3} = 1$ *at points* A *and* B. *If* $\overrightarrow{PQ} = t\overrightarrow{QA} = (2-t)\overrightarrow{QB}$, *then* $x_0 =$ ___.

Answer: $\frac{\sqrt{2}}{2}$
Reasoning: Let $l_{PQ} : y = kx + 4 (k < 0)$, $A(x_1, y_1)$. Then $Q(\frac{4}{-k}, 0)$. From $\overrightarrow{PQ} = t\overrightarrow{QA} \Rightarrow (-\frac{4}{k}, -4) = t(x_1 + \frac{4}{k}, y_1) \Rightarrow -\frac{4}{k} = t(x_1 + \frac{4}{k}), -4 = ty_1 \Rightarrow x_1 = -\frac{4}{kt} - \frac{4}{k}, y_1 = -\frac{4}{t}$. From point A being on the hyperbola, we get $(48 - 3k^2)t^2 + 96t - 16k^2 + 48 = 0$. Similarly, from $\overrightarrow{PQ} = (2-t)\overrightarrow{QB}$, we can obtain the equation $(48 - 3k^2)(2-t)^2 + 96(2-t) - 16k^2 + 48 = 0 \Rightarrow t + (2-t) = -\frac{96}{48-3k^2} \Rightarrow k = -4\sqrt{2}, x_0 = \frac{\sqrt{2}}{2}$.

Problem 60. *Assuming sequence* F_n *satisfying:* $F_1 = F_2 = 1, F_{n+1} = F_n + F_{n-1} (n \geq 2)$. *Then the number of sets of positive integers* (x, y) *that satisfy* $5F_x - 3F_y = 1$ *is*

Answer: 3
Reasoning: From the given conditions, we know for any $n \geq 2$, we have $F_{n+1} > F_n$. Notice that $F_n \in Z_+, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, \cdots$. When $x = 1, 2$, there does not exist F_r satisfying $5F_x - 3F_y = 1$. When $x = 3$, in this case, to satisfy $5F_3 - 3F_y = 1$, then $F_y = 3$, which means $y = 4$. Thus, $(x, y) = (3, 4)$ meets the requirement. By $5F_x - 3F_y = 1$, we know $y > x$. If $x + 1$=y, then simplifying ① gets $F_{x-2} - F_{x-3} = 1 (x \geq 4)$. Therefore, $x - 2 = 3$ or $4 \Rightarrow x = 5or6$. Thus, $(x, y) = (5, 6)$ or $(6, 7)$ meets the requirement. If $y = x - 2$, then $5F_x - 3F_y < 5F_x - 6F_x < 0$, it's a contradiction. Overall, $(x, y) = (3, 4)$ or (5,6) or (6,7), a total of 3 sets.

Problem 61. *For some positive integers n, there exists a positive integer $k \geq 2$ such that for positive integers $x_1, x_2, ..., x_k$ satisfying the given condition, $\sum_{i=1}^{k-1} x_i x_{i+1} = n$, $\sum_{i=1}^{k} x_i = 2019$ the number of such positive integers is _____.*

Answer: 1017073
Reasoning: $n = \sum_{i=1}^{k-1} x_i x_{i+1} (x_1+x_3+\cdots)(x_2+x_4+\cdots) = (x_1+x_3+\cdots)(2019-x_1-x_3-\cdots)$. From $1009 \times 1010 = 1019090 \Rightarrow n = 1019090$. When $x_1 = 1009$, $x_2 = 1010$, if $k = 2$, one can obtain $n = 1019090$. Let x_s be the smallest number among all considered values, then equality $n = x_s(\sum_{i=1}^{k} x_i - x_s) = x_s(2019 - x_s) = 2018$ holds true if and only if $x_1 = x_2 = \cdots = x_{2019} = 1$. Let $S = \{x|x \in Z, 1008 \times 1011x < 1009 \times 1010\}$. We will prove the range of n falls in S in the following. By dividing S into 1008 intervals: $S_{2018} = \{x|x \in Z, 1008 \times 1011 < x < 1009 \times 1010\}$, $S_i = \{x|x \in Z, i(2019-i) < x < (i+1)(2018-i)\}$, where, $i = 1, 2, \cdots, 1008$. When $n = 1009 \times 1010$, the construction is given. If $t \in S_i$ and $t \neq 1009 \times 1010$, let $t = (i+1)(2018-i) - a$, $a \in [1, 2018-2i]$, take $k = 4$, $x_1 = 1$, $x_2 = 2018 - i - a$, $x_3 = i$, $x_4 = a$, now, $n = (i+1)(2018-i) - a$. Therefore, it proved that every number in set S has corresponding k and x_i meets the problem's criteria. Thus, the sought positive number $n \in [2018, 1019090]$.

Problem 62. *Considering all non-increasing functions $f : \{1, 2, \cdots, 10\} \to \{1, 2, \cdots, 10\}$, some of these functions have fixed points, while others do not. The difference in the number of these two types of functions is _____.*

Answer: 4862
Reasoning: Below, a stronger conclusion is proven: For positive integers n, considering all non-increasing functions $f : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$, among these functions, it's demonstrated that the difference in the number of functions with and without fixed points is $C_{2n-2}^{n-1} - C_{2n-2}^{n-2} = \frac{1}{n} C_{2n-2}^{n-1}$. It's noted that there can be at most one fixed point in function f. First, using the method of inserting dividers, the number of non-increasing functions is $C_{n-1+n}^{n-1} = C_{2n-1}^{n-1}$. If a function f has a fixed point, i.e., there exists c, such that $f(c) = c$. When the fixed point is c, dividing it into two parts $[1, c-1]$ and $[c+1, n]$ and applying the method of inserting dividers again to calculate the number of non-increasing functions, the number of such functions f with a fixed point is obtained as $C_{n-c+c-1-1}^{c-1} C_{c-1+n-c+1}^{n-c} = (C_{n-1}^{c-1})^2$. Consequently, the total number of functions f with a fixed point is calculated as $\sum_{c=1}^{n} (C_{n-1}^{c-1})^2 = C_{2n-2}^{n-1}$. As a result, the number of functions f without a fixed point is found as $C_{2n-1}^{n-1} - C_{2n-2}^{n-1} = C_{2n-2}^{n-2}$. Therefore, the sought difference is calculated as $C_{2n-2}^{n-1} - C_{2n-2}^{n-2} = \frac{1}{n} C_{2n-2}^{n-1}$. In this problem, with $n = 10$, the answer is 4862.

Problem 63. *Given an integer coefficient polynomial $P(x)$ satisfying: $P(-1) = -4, P(-3) = -40, P(-5) = -156$. The maximum number of solutions x for $P(P(x)) = x^2$ is _____.*

Answer: 0
Reasoning: Notice that, $3|(P(x+3) - P(x))(x \in Z)$. If $x \equiv 0(mod3)$,then $x^2 \equiv P(P(x)) \equiv P(P(-3)) = P(-40) \equiv P(-1) = -4 \equiv -1(mod3)$, contradiction. If $x \equiv 1(mod3)$, then

$x^2 \equiv P(P(x)) \equiv P(P(-5)) = P(-156) \equiv P(-3) = -40 \equiv -1(mod3)$, contradiction. If $x \equiv 2(mod3)$, then $x^2 \equiv P(P(x)) \equiv P(P(-1)) = P(-4) \equiv P(-1) = -4 \equiv -1(mod3)$, contradiction. So the number of x satisfying $P(P(x)) = x^2$ is 0.

Problem 64. *Given hyperbola $\Gamma : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ passes the point $M(3, \sqrt{2})$, line l passes its right focus $F(2, 0)$ and cross the right branch of Γ at points A and B, and cross the y-axis at point P. If $\overrightarrow{PA} = m\overrightarrow{AF}, \overrightarrow{PB} = n\overrightarrow{BF}$, then $m + n =$ _____.*

Answer: 6
Reasoning: From the condition given, it is easy to get the equation of hyperbola Γ is $\frac{x^2}{3} - y^2 = 1$. Let $A(x_1, y_1)$, $B(x_2, y_2)$, $P(0, t)$, from $\overrightarrow{PA} = m\overrightarrow{AF} \Rightarrow x_1 = \frac{2m}{m+1}, y_1 = \frac{t}{m+1} \Rightarrow (\frac{2m}{m+1})^2 - 3(\frac{t}{m+1})^2 = 3 \Rightarrow m^2 - 6m - 3(t^2 + 1) = 0$. Similarly, from $\overrightarrow{PB} = n\overrightarrow{BF}$, we get $n^2 - 6n - 3(t^2 + 1) = 0$. Therefore, $m, n(m \neq n)$ are two real roots of the equation $x^2 - 6x - 3(t^2 + 1) = 0$. Thus, $m + n = 6$.

Problem 65. *Let positive real numbers x_1, x_2, x_3, x_4 satisfying $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 = x_1x_3 + x_2x_4$. Then the minimum of $f = \frac{x_1}{x_2} + \frac{x_2}{x_3} + \frac{x_3}{x_4} + \frac{x_4}{x_1}$ is _____.*

Answer: 8
Reasoning: From $x_1, x_2, x_3, x_4 \in R$, using the mean inequality $f = \frac{x_1}{x_2} + \frac{x_2}{x_3} + \frac{x_3}{x_4} + \frac{x_4}{x_1} \geq 2\sqrt{\frac{x_1x_3}{x_2x_4}} + 2\sqrt{\frac{x_2x_4}{x_1x_3}} = \frac{2(x_1x_3+x_2x_4)}{\sqrt{x_1x_3x_2x_4}} = \frac{2(x_1+x_3)(x_2+x_4)}{\sqrt{x_1x_2x_3x_4}} \geq \frac{8\sqrt{x_1x_3}\sqrt{x_2x_4}}{\sqrt{x_1x_2x_3x_4}} = 8$. The equality holds when $x_1 = x_3 = 1$, $x_2 = x_4 = 2 + \sqrt{3}$.

Problem 66. *Given sequence $\{a_n\}$ satisfying $a_1 = a, a_{n+1} = 2(a_n + \frac{1}{a_n}) - 3$. If $a_{n+1} > a_n(n \in Z_+)$. The range of real number a is _____.*

Answer: $(0, \frac{1}{2}) \cup (2, +\infty)$
Reasoning: From $a_2 - a_1 = \frac{(a-1)(a-2)}{a} > 0 \Rightarrow 0 < a < 2ora > 2$. (1) When $a > 2$, from the induction we can prove $a_n > 2 \Rightarrow a_{n+1} - a_n = \frac{(a_n-1)(a_n-2)}{a_n} > 0 \Rightarrow a_{n+1} > a_n$. (2) When $0 < a < \frac{1}{2}$, $a_2 = 2(a + \frac{1}{2}) - 3 > 2 \Rightarrow a_n > 2 \Rightarrow a_{n+1} > a_n > a_1(n \geq 2)$. When $\frac{1}{2} < 1 < 1$, $a_2 = 2(a + \frac{1}{a}) - 3 \in (1, 2] \Rightarrow a_n \in (1, 2] \Rightarrow a_{n+1} - a_n < 0(n \geq 2)$, which does not satisfy the requirement. From the above all, $a \in (0, \frac{1}{2} \cup (2, +\infty))$

Problem 67. *Given positive number $\alpha, \beta, \gamma, \delta$ satisfying $\alpha + \beta + \gamma + \delta = 2\pi$, and $k = \frac{3\tan\alpha}{1+\sec\alpha} = \frac{4\tan\beta}{1+\sec\beta} = \frac{5\tan\gamma}{1+\sec\gamma} = \frac{6\tan\delta}{1+\sec\delta}$, then k= _____.*

Answer: $\sqrt{19}$
Reasoning: From the given condition, we can obtain $k = 3\tan\frac{\alpha}{2} = 4\tan\frac{\beta}{2} = 5\tan\frac{\gamma}{2} = 6\tan\frac{\delta}{2}$. Let $a = \tan\frac{\alpha}{2}$, $b = \tan\frac{\beta}{2}$, $c = \tan\frac{\gamma}{2}$, $d = \tan\frac{\delta}{2}$. Then $0 = \tan\frac{\alpha+\beta+\gamma+\delta}{2} = \frac{a+b+c+d-abc-bcd-cda-dab}{1+abcd-ab-ac-ad-bc-bd-cd} \Rightarrow k^3 - 19k = 0 \Rightarrow k = \sqrt{19}$ ($k = 0, \sqrt{19}$ is abandoned).

Problem 68. *Let $A = \{1, 2, \cdots, 6\}$, function $f : A \to A$. Mark $p(f) = f(1) \cdots f(6)$. Then the number of functions that make $p(f)|36$ is _____.*

Answer: 580
Reasoning: Because $p(f)|36$, so $p(f)|2^a3^b$, $a, b \in \{0, 1, 2\}$. We will count by category in the

following.
(1) If $b=0$, then the number of choices for a can be $C_6^0(a=0)$, $C_6^1(a=1), C_6^1+C_6^2(a=2)$, where, $a=2$, the two 2 are in different or the same among $f(1),\cdots,f(6)$.
(2) If $b=1$, then there are C_6^1 choices for 3. The choices for a can be $C_6^0(a=0), C_6^1(a=1), C_5^1+C_6^2(a=2)$, where, $a=2$, the 2 are among the different or same $f(1),\cdots,f(6)$ but not in the one that contains 3.
(3) If $b=2$, then there are C_6^2 choices for 3, the choices of a can be $C_6^0(a=0)$, $C_6^1(a=1)$, $C_4^1+C_6^2(a=2)$, where, $a=2$, the two 2 can be in different or same among $f(1),\cdots,f(6)$, but can not be in the two that contains 3. Therefore, in total, $(C_6^0+2C_6^1+C_6^2)+C_6^1(C_6^0+C_6^1+C_5^1+C_6^2)+C_6^2(C_6^0+C_6^1+C_4^1+C_6^2)=580$

Problem 69. *If unit complex number a,b satisfy $a\bar{b}+\bar{a}b=\sqrt{3}$, then $|a-b|=$ _____.*

Answer: $\frac{\sqrt{6}-\sqrt{2}}{2}$
Reasoning: From $|a-b|^2=(a-b)(\bar{a}-\bar{b})=1-a\bar{b}-\bar{a}b+1=2-\sqrt{3}\Rightarrow|a-b|=\sqrt{2-\sqrt{3}}=\frac{\sqrt{6}-\sqrt{2}}{2}$

Problem 70. *The right focus F_1 of the ellipse $\Gamma_1:\frac{x^2}{24}+\frac{y^2}{b^2}=1(0<b<2\sqrt{6})$ coincides with the focus of the parabola $\Gamma_2: y^2=4px(p\in Z_+)$. The line l passing through the point F_1 with a positive integer slope intersects the ellipse Γ_1 at points A and B, and intersects the parabola Γ_2 at points C and D. If $13|AB|=\sqrt{6}|CD|$, then $b^2+p=$ _____.*

Answer: 12
Reasoning: Assume line $l:k(x-p(k\in Z_+)$, combine $y^2=4px$, we obtain $k^2x^2-2p(k^2+2)x+k^2p^2=0$. Assume $C(x_1,y_1)$, $D(x_2,y_2)$, then $x_1+x_2=-\frac{2p(k^2+2)}{k^2}, x_1x_2=p^2$. So $|CD|^2=\frac{16p^2(1+k^2)^2}{k^4}$. Combine $y=k(x-p)$ and $\frac{x^2}{24}+\frac{y^2}{b^2}=1$, we obtain $(b^2+24k^2)x^2-48pk^2x+24(k^2p^2-b^2)=0$. Assume $A(x_3,y_3)$, $B(x_4,y_4)$, then $x_3+x_4=-\frac{48pk^2}{b^2+24k^2}$, then $x_3x_4=\frac{24(k^2p^2-b^2)}{b^2+24k^2}$, so $|AB|^2=\frac{96b^2(1+k^2)(b^2+24k^2-p^2k^2)}{(b^2+24k^2)^2}$. From $13|AB|=\sqrt{6}|CD|$ and $b^2=24-p^2$, we obtain $13k^2(24-p^2)=p(24-p^2+24k^2)$. From $p^2<24$, p is positive integer, only $p=4$ satisfies the condtion, therefore, $b=2\sqrt{2}$. Thus, $b^2+p=8+4=12$.

Problem 71. *Given that $p(x)$ is a quintic polynomial. If $x=0$ is a triple root of $p(x)+1=0$ and $x=1$ is a triple root of $p(x)-1=0$, then the coefficient of the x^3 term in the expression of $p(x)$ is _____.*

Answer: 20
Reasoning: Let $p(x)+1=x^3(ax^2+bx+c)$, $p(x)-1=(x-1)^3(lx^2+mx+n)$ Differentiate the above equation to the first and second order, respectively, and set $x=0$, then compare the coefficients of the corresponding equations to solve for the coefficient. We get $l=12, m=6, n=2$. Therefore, $p(x)=12x^5-30x^4+20x^3-1$.

Problem 72. *Set $X=\{1,2,\cdots,10\}$, mapping $f:X\to X$ satisfy:*
(1) $f\circ f=I_x$, where, $f\circ f$ is a composite mapping, I_x is an identity mapping on X.
(2) $|f(i)-i|=2$, for any $i\in X$.
Then the number of mapping f is _____.

Answer: 401
Reasoning: From (1), we know (i) $f(x) = x$, (ii) $f(x) = y(x \neq y)$, $f(y) = x$. Set the number of mapping f_n satisfying the condition when $S_n = \{1, 2, \cdots, n\}$. When $f(1) = 1$, the number of mapping is f_{n-1}. When $f(1) = 2$, then $f(2) = 1$, the number of mapping is f_{n-2}. When $f(1) = 3$, then $f(3) = 1$, if $f(2) = 2$, the number of mapping is f_{n-3}. If $f(2) = 4$, then $f(4) = 2$, the number of mapping is f_{n-4}. Therefore, $f_n = f_{n-1} + f_{n-2} + f_{n-3} + f_{n-4}$. By calculation, $f_1 = 1$, $f_2 = 2$, $f_3 = 4$, $f_4 = 8$, $\cdots$, $f_{10} = 401$.

Problem 73. *Given a integer coefficient polynomial of degree 2022 with leading coefficient is 1, how many roots can it possibly have in the interval (0,1) as maximum?*

Answer: 2021
Reasoning: First, if all 2022 roots of the polynomial are within the interval (0,1), then according to Vieta's formulas, its constant term is the product of these 2022 roots, which also must lie within the interval, thus it cannot be an integer, which is a contradiction. Therefore, the polynomial can have at most 2021 roots in the interval $(0, 1)$.

Next, we prove that there exists a leading coefficient 1 integer coefficient polynomial of degree 2022, which has at least 2021 roots in the interval $(0, 1)$. Let $P(x) = x^2022 + (1 - 4042x)(3 - 4042x) \cdot (5 - 4042x) \cdots (4041 - 4042x)$. Note that, for each $k = 0, 1, \cdots, 2021$, we have $P(\frac{2k}{4042}) = (\frac{2k}{4042})^{202} + (-1)^k (2k - 1)!! \cdot (4041 - 2k)!!$. When k is even, its value is positive; when k is odd, its value is negative. It is evident that there are at least 2021 sign changes in the interval $(0, 1)$, therefore, it has at least 2021 roots in the interval.

Problem 74. *The system of equations* $\begin{cases} x^2y + y^2z + z^2 = 0, \\ z^3 + z^2y + zy^3 + x^2y = \frac{1}{4}(x^4 + y^4) \end{cases}$ *has how many sets of real solution* (x, y, z) *?.*

Answer: 1
Reasoning:
First consider the case that x, y, z are all not equal to 0.
Set $y = kz$. then the system of equations can be written

$$\begin{cases} \frac{x^2}{y} + z + \frac{z^2}{y^2} = 0 \\ \frac{4z^3}{y^2} + \frac{4z^2}{y} + 4yz + \frac{4x^2}{y} = \frac{x^4}{y^2} + y^2 \end{cases}$$

$$\begin{cases} \frac{x^2}{y} = -z - \frac{z^2}{y^2}, \\ \left(\frac{x^2}{y} - 2\right)^2 = 4yz + \frac{4z^2}{y} + \frac{4z^3}{y^2} - y^2 + 4 \end{cases} \Rightarrow \left(z + \frac{z^2}{y^2} + 2\right)^2 = 4yz + \frac{4z^2}{y} + \frac{4z^3}{y^2} - y^2 + 4.$$

put $y = kz$ into the above equation

$$\left(z + \frac{1}{k^2} + 2\right)^2 = 4kz^2 + \frac{4z}{k^2} + \frac{4z}{k} - k^2z^2 + 4$$
$$\Rightarrow (4k - k^2 - 1)z^2 + \left(\frac{2}{k^2} + \frac{4}{k} - 4\right)z - \left(\frac{1}{k^4} + \frac{4}{k^2}\right) = 0.$$

Notice that,

$$\Delta = \left(\frac{2}{k^2} + \frac{4}{k} - 4\right)^2 + 4\left(\frac{1}{k^4} + \frac{4}{k^2}\right)(4k - k^2 - 1)$$
$$= \frac{32}{k^3} + \frac{32}{k} - \frac{20}{k^2}.$$

If $k < 0$, i.e. y, z have different signs, then, $\Delta < 0, z$ has no solution. Thus, the system of equations has no solution.
If $k > 0$, when $y > 0, z > 0$, $\frac{x^2}{y} + z + \frac{z^2}{y^2} > 0$, is contradiction with (1). Thus, $y < 0, z < 0$.
Solved from the original system of equations $z^3 - x^2y^2 + x^2y = \frac{1}{4}(x^4 + y^4)$

now, $z^3-x^2y^2+x^2y<0, \frac{1}{4}\left(x^4+y^4\right)>0$, contradiction. Therefore, there must be 0 among x, y, z.
(1) When $z=0$, the system of equations becomes $x^4+y^4=0 \Rightarrow x=y=0$. Thus, $x=y=z=0$.
(2) When $y=0$, $\left\{\begin{array}{l} z^2=0 \\ z^3=\frac{1}{4}x^4 \end{array} \Rightarrow x=y=z=0\right.$.
(3) When $x=0$, $\left\{\begin{array}{l} z\left(y^2+z\right)=0, \\ z^3+zy\left(z+y^2\right)=\frac{1}{4}y^4 \end{array} \Rightarrow z^3=\frac{1}{4}y^4\right.$. When y, z are not equal to 0 at the same time, then $z>0$. Therefore, $y^2+z=0$, contradiction.Thus, $y=z=0$.
Therefore, the only solution of the system of equations is $(x, y, z)=(0,0,0)$.

Problem 75. *Set x, y, z are real numbers, satisfying $x^2+y^2+z^2=1$. Then the maximum and minimum of $(x-y)(y-z)(x-z)$ are*

Answer: $\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}$
Reasoning: Notice that, for any arrangement of x, y, z, only change the sign of the required formula, don't change the absolute value of it. Thus, only the maximum needs to be calculated. Set $x \geq y \geq z$. Then, from the mean inequality,

$$(x-y)(y-z) \leq\left(\frac{(x-y)+(y-z)}{2}\right)^2=\left(\frac{x-z}{2}\right)^2 \Rightarrow(x-y)(y-z)(x-z) \leq \frac{(x-z)^3}{4}.$$

Thus, only need to prove $x-z \leq \sqrt{2}$. In fact, $(x-z)^2=2x^2+2z^2-(x+z)^2 \leq 2\left(x^2+z^2\right)=2-2y^2 \leq 2$, When $x=\frac{1}{\sqrt{2}}, y=0, z=-\frac{1}{\sqrt{2}}$, the equation above holds true. Hence, the maximum value sought is $\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$, the minimum value is $-\frac{1}{\sqrt{2}}=-\frac{\sqrt{2}}{2}$.

Problem 76. *If there are a total of 95 numbers, each of which can take any value in +1 or -1, and it is known that the sum of the pairwise products of these 95 numbers is positive, then what is the minimum possible value of this positive sum?*

Answer: 13
Reasoning: Let the minimum positive value be N, then we have $95+2\text{ N}=\sum_{i=1}^{95} a_i^2+2\text{ N}=\left(\sum_{i=1}^{95} a_i\right)^2$ being a perfect square, The smallest perfect square larger than 95 is 121, the corresponding N is 13, constructed with 53 +1s and 42 -1s.

Problem 77. *A monotonically increasing sequence of positive integers, starting from the third term, with each subsequent term being the sum of the preceding two terms. If its seventh term is 120, then its eighth term is ______.*

Answer: 194
Reasoning: Set first two terms as $x<y$, Then, from the given conditions, it can be deduced that $5x+8y=120$, restricting $x<y$, the only positive integer solution is $x=8, b=10$, hence, it can be further determined that the eighth term is $8x+13y=194$.

Problem 78. *Sequence $\{a_n\}$ satisfy $a_0=0, a_1=1$, and for any positive integer* n, *we have $a_{2n}=a_{n'} a_{2n+1}=a_n+1$, then $a_{2024}=$______.*

Answer: 7

Reasoning: $a_{2024} = a_{1012} = a_{506} = a_{253} = a_{126} + 1 = a_{63} + 1 = a_{31} + 2 = a_{15} + 3 = a_7 + 4 = a_3 + 5 = a_7 + 6 = 7$, in fact a_n is n, which is the number of ones in its binary representations.

Problem 79. *Non-negative real numbers* x, y, z *satisfy* $4x^2 + 4y^2 + z^2 + 2z = 3$, *then the minimum of* $5\mathrm{x} + 4\mathrm{y} + 3\mathrm{z}$ *is* ____.

Answer: 3

Reasoning: The given conditions, when squared, result in $(2x)^2 + (2y)^2 + (z+1)^2 = 4$, Combined with the non-negative condition, it can be obtained that $\mathrm{x}, \mathrm{y}, \mathrm{z} \in [0, 1]$, Furthermore, by utilizing range scaling, we have $5x + 4y + 3z \geq 4x^2 + 4y^2 + 3z \geq 4x^2 + 4y^2 + z^2 + 2z = 3$.

Problem 80. *Real numbers* x, y *satisfy* $2x - 5y \leq -6$ *and* $3x + 6y \leq 25$, *then the maximum value of* $9x + y$ *is* ____.

Answer: $\frac{869}{27}$

Reasoning: By adjusting coefficients, we can obtain $27(9x+y) = 47(3x+6y) + 51(2x-5y) \leq 47 \times 25 + 51 \times (-6) = 869$.

Problem 81. *The sum of the maximum element and minimum element in set* $\left\{ \frac{3}{a} + b \middle| \, 1 \leq a \leq b \leq 2 \right\}$ *is* ____.

Answer: $5 + 2\sqrt{3}$

Reasoning: The maximum element needs to make a as small as possible, b as large as possible, therefore it is obvious $3 + 2 = 5$, while the minimum element needs to make a as large as possible, b as small as possible, therefore, we have $a = \mathrm{b}$, and utilizing the mean inequality we select $a = \mathrm{b} = \sqrt{3}$, the the minimum element is $2\sqrt{3}$.

Problem 82. *A monotonically increasing function* f(x) *defined on* R^+ *satisfies* $\mathrm{f}\left(\mathrm{f(x)} + \frac{2}{\mathrm{x}}\right) = -1$ *consistently holds within its domain, then* f(1) =____.

Answer: -1

Reasoning: According to the given conditions, it can only be $\mathrm{f(x)} + \frac{2}{x}$ is a constant value, set this constant to be $\mathrm{t} > 0$, then we have $\mathrm{f(t)} + \frac{2}{t} = t$ and $\mathrm{f(t)} = -1$, solved as $\mathrm{t} = 1$ (neglecting the negative root), therefore $\mathrm{f(1)} = -1$.

Problem 83. *Given set:* $A = \left\{ x + y \middle| \, \frac{x^2}{9} + y^2 = 1 \, , x + y \in \mathbf{Z}_+ \right\}$, $B = \{2x + y \mid x, y \in A, x < y\}$, $C = \{2x + y \mid x, y \in A, x > y\}$. *Then the sum of all elements in* $B \cap C$ *is*

Answer: 12

Reasoning: Set $x = 3\cos\theta, y = \sin\theta$. Then $x + y = \sqrt{10}\sin(\theta + \varphi)$. Thus, $A = \{1, 2, 3\}$. Therefore, through enumeration we obtain $B = \{4, 5, 7\}, C = \{5, 7, 8\} \Rightarrow B \cap C = \{5, 7\}$. So, $5 + 7 = 12$.

Problem 84. *Given a hyperbola* $\Gamma : \frac{x^2}{7} - \frac{y^2}{5} = 1$, *a line* $l : ax + by + 1 = 0$ *intersects* Γ *at point* A. *A tangent to* Γ *drawn through point* A *is perpendicular to the line* l. *Then* $\frac{7}{a^2} - \frac{5}{b^2} =$____.

Answer: 144

Reasoning: Set point $A(x_0, y_0)$. Then the tangent is $\frac{x_0 x}{7} - \frac{y_0 y}{5} = 1$, slope is $\frac{5x_0}{7y_0}$. Additionally, the slope of l is $-\frac{a}{b}$, then from the given condition $\frac{5ax_0}{7b} y_0 = 1 \Rightarrow x_0 = \frac{7by_0}{5a}$. Combining point A is on line l, we have $x_0 = -\frac{b}{a} y_0 - \frac{1}{a}$. Therefore, $y_0 = -\frac{5}{12b}, x_0 = -\frac{7}{12a}$. Substituting into the equation of the hyperbola Γ, we obtain $\frac{7}{144a^2} - \frac{5}{144b^2} = 1 \Rightarrow \frac{7}{a^2} - \frac{5}{b^2} = 144$.

Problem 85. *In the Cartesian coordinate plane xOy, point $A(a,0), B(0,b), C(0,4)$, moving point D satisfies $|CD| = 1$. If the maximum value of $|\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OD}|$ is 6, then the minimum value of $a^2 + b^2$ is _____.*

Answer: 1

Reasoning: From moving point D satisfying $|CD| = 1$, we know point D lies on a circle with center C and radius 1. Set $D(x, y)$, then $|\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OD}| = \sqrt{(x+a)^2 + (y+b)^2}$, i.e., the maximum value of distance between point (x, y) and point $(-a, -b)$ is 6, this indicates that point $(-a, -b)$ lies on a circle with center C and radius 5. So $a^2+b^2 = (-a)^2+(-b)^2 \geq 1$, when $a = 0, b = 1$, the equation holds true.

Problem 86. *Given n is positive integer, for $i = 1, 2, \cdots, n$, positive integer a_i and positive even number b_i satisfy $0 < \frac{a_i}{b_i} < 1$, and for any positive integers $i_1, i_2\,(1 \leq i_1 < i_2 \leq n), a_{i_1} \neq a_{i_2}$ and $b_{i_1} \neq b_{i_2}$ at least one holds true. If for any positive integer n and all positive integers a_i and positive even numbers b_i satisfy the above conditions, we all have $\frac{\sum_{i=1}^n b_i}{n^{\frac{3}{2}}} \geq c$, then the maximum value of real number c is _____.*

Answer: $\frac{4}{3}$

Reasoning: For any positive integer t, when $n = t^2$, take integers that satisfy conditions:

$a_1 = 1, b_1 = 2; a_2 = 1, b_2 = 4;$

$a_3 = 2, b_3 = 4; a_4 = 3, b_4 = 4;$

$\cdots$

$a_{(t-1)^2+1} = 1, b_{(t-1)^2+1} = 2t;$

$a_{(t-1)^2+2} = 2, b_{(t-1)^2+2} = 2t;$

$a_{t^2} = 2t-1, b_{t^2} = 2t.$

Then $\sum_{i=1}^n b_i = \sum_{j=1}^t (2j-1)2j = 4\sum_{j=1}^t j^2 - 2\sum_{j=1}^t j = \frac{2t(t+1)(2t+1)}{3} - t(t+1) = \frac{t(t+1)(4t-1)}{3}$.

So, $c \leq \frac{\sum_{i=1}^n b_i}{n^{\frac{1}{2}}} = \frac{t(t+1)(4t-1)}{3(t^2)^{\frac{3}{2}}} = \frac{(t+1)(4t-1)}{3t^2} \to \frac{4}{3}(t \to +\infty)$.

Therefore, $c \leq \frac{4}{3}$.

The following is proved by mathematical induction: for each positive integer n and all positive integers a_i satisfying the conditions and all positive even numbers $b_i (i = 1, 2, \cdots, n)$, we all have $\frac{\sum_{i=1}^n b_i}{n^{\frac{3}{2}}} \geq \frac{4}{3}$.

When $n = 1$, since $b_1 \geq 2$, then $\frac{b_1}{1^{\frac{3}{2}}} \geq 2 > \frac{4}{3}$.

Assume that equation (1) holds true, when $n = k$ (k is positive integer). When $n = k+1$, consider positive integers a_i and positive even numbers $b_i (i = 1, 2, \cdots, k+1)$ satisfying conditions. Next, we aim to prove that there exists $i_0 \in \{1, 2, \cdots, k+1\}$, such that $b_{i_0} \geq 2\sqrt{k+1}$.

Assume that the conclusion is not true, then for each $i = 1, 2, \cdots, k+1$, we have $2 \leq b_i < 2\sqrt{k+1}$. Without loss of generality, suppose that the sequence $\{b_i\}\,(i = 1, 2, \cdots, k+1)$ is

arranged in increasing order. Set $\sqrt{k+1} = m + a$, where, $m = [\sqrt{k+1}]$ ($[x]$ denotes the greatest integer not exceeding x), $0 \le a < 1$. Then $2 \le b_i < 2\sqrt{k+1} = 2m+2a < 2m+2 \Rightarrow 2 \le b_i \le 2m+1$. Since b_i is even number, thus, $2 \le b_i \le 2m$.
For positive integer $j(1 \le j \le m)$, the number of positive integers a_i satifying $b_i = 2j$ is at most $2j-1$, i.e., $a_i \in \{1, 2, \cdots, 2j-1\}$. Therefore, $n = k+1 \le \sum_{j=1}^{m}(2j-1) = m^2$.
Thus, $m \ge \sqrt{k+1} \ge m$, which implies that all equalities hold, i.e., $m = \sqrt{k+1}$, $n = k+1 = \sum_{j=1}^{m}(2j-1) = m^2$, $b_{k+1} = 2\sqrt{k+1}$, contradiction. Therefore, there exists $i_0 \in \{1, 2, \cdots, k+1\}$, such that $b_{i_0} \ge 2\sqrt{k+1}$.
From the induction hypothesis, we can obtain

$$\sum_{i=1}^{k+1} b_i = \sum_{\substack{i=1 \\ i \ne i_0}}^{k+1} b_i + b_{i_0} > \frac{4}{3}k^{\frac{3}{2}} + 2\sqrt{k+1}.$$

To prove that equation (1) holds when n=k+1$itsufficestoshow\frac{4}{3}k^{\frac{3}{2}} + 2\sqrt{k+1} \ge \frac{4}{3}(k+1)^{\frac{2}{2}}$.
Notice that, equation (2)

$$\Leftrightarrow \frac{4}{3}k\sqrt{k} + 2\sqrt{k+1} \ge \frac{4}{3}(k+1)\sqrt{k+1}$$
$$\Leftrightarrow 2k\sqrt{k} + 3\sqrt{k+1} \ge 2(k+1)\sqrt{k+1}$$
$$\Leftrightarrow \sqrt{k+1} \ge 2k(\sqrt{k+1} - \sqrt{k}).$$

From $2k(\sqrt{k+1} - \sqrt{k}) = \frac{2k}{\sqrt{k+1}+\sqrt{k}} < \frac{k}{\sqrt{k}} = \sqrt{k} < \sqrt{k+1}$.
Then equation (3) holds, thus equation (2) holds, thereby completing the proof of equation (1). From equation (1), we know that when $c = \frac{4}{3}$, the original inequality holds.
Therefore, $c_{\max} = \frac{4}{3}$.

Problem 87. *In a cube $ABCD - A_1B_1C_1D_1$, $AA_1 = 1$, E, F are the midpoints of edges CC_1, DD_1, then the area of the cross-section obtained by the plane AEF intersecting the circumscribed sphere of the cube is _____.*

Answer: $\frac{7}{10}\pi$
Reasoning: Taking A as the origin, and AB, AD, AA_1 as the x, y, z axes to establish a spatial rectangular coordinate system, then, $A(0,0,0)$,$E(1,1,\frac{1}{2})$,$F(0,1,\frac{1}{2})$, so $\overrightarrow{AE} = (1,1,\frac{1}{2})$, $\overrightarrow{AF} = (0,1,\frac{1}{2})$. Let the normal vector of plane AEF be $n = (x,y,z)$. Then,

$$\begin{cases} \mathbf{n} \cdot \mathbf{AE} = 0 \\ \mathbf{n} \cdot \mathbf{AF} = 0 \end{cases} \Rightarrow \begin{cases} x + y + \dfrac{1}{2}z = 0, \\ \qquad y + \dfrac{1}{2}z = 0. \end{cases}$$

Take $n = (0,-1,2)$, the distance from the center of the sphere $O(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ to the plane AEF is $d = \frac{|\overrightarrow{AO}| \cdot n}{|n|} = \frac{\sqrt{5}}{10}$. Let the radius of the cross-sectional circle be r, because the radius of the circumscribed sphere of the cube is $R = \frac{\sqrt{3}}{2}$, therefore,$r^2 = R^2 - d^2 = \frac{3}{4} - \frac{1}{20} = \frac{7}{10}$, thus the area of the cross-section is $\frac{7}{10}\pi$.

Problem 88. *In tetrahedron $ABCD$, triangle ABC is an equilateral triangle, $\angle BCD = 90°$, $BC = CD = 1$,$AC = \sqrt{3}$,E and F are the midpoints of edges BD and AC respectively. Then the cosine of the angle formed by lines AE and BF is _____.*

Answer: $\frac{\sqrt{2}}{3}$
Reasoning: Take the midpoint M of CE, then $\angle MFB$ is the angle formed by AE and BF, where $FM = \frac{1}{2}AE = \frac{\sqrt{6}}{4}$. Hence, $cos\angle MFB = \frac{\frac{3}{16}+\frac{3}{4}-\frac{10}{16}}{2\times\frac{\sqrt{6}}{4}\times\frac{\sqrt{3}}{2}} = \frac{\sqrt{2}}{3}$.

Problem 89. *Let P be a point inside triangle ABC, and $2\overrightarrow{PA} = \overrightarrow{PB} + \overrightarrow{PC} = 0$. If $\angle BAC = \frac{\pi}{3}, BC = 2$, then the maximum value of $\overrightarrow{PB} \cdot \overrightarrow{PC}$ is _____.*

Answer: $-\frac{1}{4}$

Reasoning: Let M be the midpoint of BC. Then $2\overrightarrow{PB} = \overrightarrow{PB} + \overrightarrow{PC} = -2\overrightarrow{PA}$, hence, P is the midpoint of the median AM of edge BC. Therefore, $\overrightarrow{PB} \cdot \overrightarrow{PC} = (\overrightarrow{PM} + \overrightarrow{MB}) \cdot (\overrightarrow{PM} + \overrightarrow{MC}) = |\overrightarrow{PM}|^2 + \overrightarrow{MB} \cdot \overrightarrow{MC} = |\overrightarrow{PM}|^2 - 1 = \frac{1}{4}|\overrightarrow{AM}|^2 - 1$. Also, since $\angle BAC = \frac{\pi}{3}$, and $BC = 2$, the locus of point A is a circle, where BC is a chord on the circle, and the inscribed angle corresponding to BC is 60°, then $AM \leq \sqrt{3}$. Therefore, $\overrightarrow{PB} \cdot \overrightarrow{PC} = \frac{1}{4}|\overrightarrow{AM}|^2 - 1 < leq\frac{3}{4} - 1 = -\frac{1}{4}$.

Problem 90. *A rectangular solid whose length, width, and height are all natural numbers, and the sum of all its edge lengths equals its volume, is called a "perfect rectangular solid. The maximum value of the volume of a perfect rectangular solid is _____.*

Answer: 120

Reasoning: Let the length, width, and height be a, b, c, and $a > b > c > 1$. Then $4(a+b+c) = abc \Rightarrow a = \frac{4(b+c)}{bc-4} \Rightarrow bc > 4 \Rightarrow b^2 > bc > 4$. Since $a > b$, we have

$$8b > (b^2 - 4)\,c > b^2 - 4 \Rightarrow \begin{cases} b \geq 3, \\ (b-4)^2 > 20 \end{cases} \Rightarrow 3 < b < 8\,.$$

Thus $(a, b, c) = (10, 3, 2), (6, 4, 2), (24, 5, 1), (14, 6, 1), (9, 8, 1)$. Therefore, the sought maximum is $24 \times 5 \times 1 = 120$.

Problem 91. *In the convex quadrilateral $ABCD$ inscribed in a circle, if $\overrightarrow{AB} + 3\overrightarrow{BC} + 2\overrightarrow{CD} + 4\overrightarrow{DA} = 0$, and $|\overrightarrow{AC}| = 4$, then the maximum of $|\overrightarrow{AB}| + |\overrightarrow{BC}|$ is _____.*

Answer: $4\sqrt{2}$

Reasoning: Set AC and BD intersects at point P. From the given conditions, $\Leftrightarrow 3\overrightarrow{PA} + \overrightarrow{PC} = 2\overrightarrow{PB} + 2\overrightarrow{PD} \Rightarrow |\overrightarrow{PA}| : |\overrightarrow{PC}| = 1 : 3$, $|\overrightarrow{PB}| : |\overrightarrow{PD}| = 1 : 1$. Furthermore, by the Power of a Point theorem $(|\overrightarrow{PA}|, |\overrightarrow{PC}|, |\overrightarrow{PB}|, |\overrightarrow{PD}|) = (1, 3, \sqrt{3}, \sqrt{3})$. Set $\angle APB = \theta$. From Cauchy's inequality: $|\overrightarrow{AB}| + |\overrightarrow{BC}| = \sqrt{4 - 2\sqrt{3}\cos\theta} + \sqrt{3(4 + 2\sqrt{3}\cos\theta)}$, $\sqrt{(1+3)(4+4)} = 4\sqrt{2}$, when $\cos\theta = \frac{\sqrt{3}}{3}$, the equality in the above equation holds.

Problem 92. *Given that the edge length of the cube $ABCD - A_1B_1C_1D_1$ is 1, where, E is the middle point of AB, F is the middle point of CC_1. Then the distance from point D to the plane passing through the three points D_1, E, F is _____.*

Answer: $\frac{4}{29}\sqrt{29}$

Reasoning: Set D as the origin, establish a three-dimensional Cartesian coordinate system respectively with DA, DC, and DD_1 as the x, y, and z axes, then $D_1(0, 0, 1), E\left(1, \frac{1}{2}, 0\right), F\left(0, 1, \frac{1}{2}\right)$. normal vector $\mathbf{n} = (3, 2, 4)$ to the plane passing through points D_1, E, F. Additionally, $\overrightarrow{DD_1} = (0, 0, 1)$, the distance from point D to the plane passing through points D_1, E, F is $\frac{|\overrightarrow{DD_1} \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{4}{29}\sqrt{29}$.

Problem 93. *Given that the vertices of triangle $\triangle OAB$ are $O(0, 0)$, $A(4, 4\sqrt{3})$, and $B(8, 0)$, with the incenter denoted as I, let Γ be a circle passing through points A and B, intersecting circle $\odot I$ at points P and Q. If the tangents drawn through points P and Q are perpendicular, then the radius of circle Γ is _____.*

Answer: $2\sqrt{7}$
Reasoning: Let $\odot I$ tangent to BO, AB, and AO at points D, E, and F, respectively. Due to $\triangle OAB$ being an equilateral triangle, $D(4,0)$, $E(6,2\sqrt{3})$, and $F(2,2\sqrt{3})$.
From the given information and the power of a point theorem, we know that circle Γ passes through the midpoints of DE and EF, denoted as $G(5,\sqrt{3})$ and $H(4,2\sqrt{3})$, respectively. Moreover, since circle Γ passes through points A and B, the radius of circle Γ is found to be $2\sqrt{7}$.

Problem 94. *A person has some $2\times5\times8$ bricks and some $2\times3\times7$ bricks, as well as a $10\times11\times14$ box. All bricks and the box are rectangular prisms. He wants to pack all the bricks into the box so that the bricks can fill the entire box. The number of bricks he can fit into the box is _____ pieces.*

Answer: 24
Reasoning: Let the number of $2\times5\times8$ bricks be a and the number of $2\times3\times7$ bricks be b. According to the problem, we have $2\times5\times8\cdot a+2\times3\times7\cdot b=10\times11\times14 \Rightarrow 40a+21b=770$. Since $(21,770)=7$ and $(40,7)=1$, we know that $7\mid a$. So, $40a\equiv 0 \pmod 7 \Rightarrow a\equiv 0\cdot\frac{77}{4}\equiv 0 \pmod 7$. This implies that $a=7$ or 14. When $a=7$, $b=\frac{490}{21}=\frac{70}{3}\notin\mathbf{Z}$; When $a=14$, $b=\frac{210}{21}=10$, satisfying the requirements. In this case, a total of $14+10=24$ bricks are used. Next, let's prove that we can indeed use 14 $2\times5\times8$ bricks and 10 $2\times3\times7$ bricks to form a $10\times11\times14$ rectangular prism. We stack 7 $2\times5\times8$ bricks vertically to form a $14\times5\times8$ rectangular prism, and stack the remaining 7 $2\times5\times8$ bricks vertically to form another $14\times5\times8$ rectangular prism. Then, we horizontally combine these two $14\times5\times8$ prisms to form a $14\times10\times8$ prism. Similarly, we stack 5 $2\times3\times7$ bricks vertically to form a $10\times3\times7$ rectangular prism, and stack the remaining 5 $2\times3\times7$ bricks vertically to form another $10\times3\times7$ rectangular prism. Then, we horizontally combine these two $10\times3\times7$ prisms to form a $10\times3\times14$ prism. Finally, we horizontally combine the $10\times8\times14$ prism with the $10\times3\times14$ prism, resulting in a $10\times11\times14$ rectangular prism, meeting the requirements. Therefore, the required number of bricks is 24.

Problem 95. *Let a be an acute angle not exceeding $45°$. If $\cot 2a-\sqrt{3}=\sec a$, then $a=$ _____ degrees.*

Answer: 10
Reasoning: According to the given conditions: $\frac{1}{\cos a}=\cot 2a-\sqrt{3}=\frac{\frac{1}{2}\cos 2a-\frac{\sqrt{3}}{2}\sin 2a}{\frac{1}{2}\sin 2a}=\frac{\sin(30°-2a)}{\sin a\cdot\cos a}\Rightarrow \sin a=\sin(30°-2a)$. Additionally, $0<a<45°$, so $a=10°$.

Problem 96. *Known that there is a regular 200-gon $A_1A_2\ldots A_{200}$, connecting the diagonals $A_iA_{i+9}(\mathrm{i}=1,2,\ldots,200)$, where $A_{i+200}=A_i(i=1,2,\ldots,9)$. Then there are a total of _____ distinct intersection points inside the regular 200-gon.*

Answer: 1600
Reasoning: Obviously, each diagonal intersects with $8\times2=16$ other diagonals. Therefore, there are a total of $200\times16\,\mathrm{div}\,2=1600$ intersections. Moreover, all these diagonals should be tangent to the same circle, which is concentric with and smaller than the circumscribed circle of the regular 200-gon. Since there can be at most two tangents passing through a point, there are no three lines intersecting at the same point.

Problem 97. *The distance between the highest point of the ellipse obtained by counterclockwise rotating the ellipse* $\frac{x^2}{2} + y^2 = 1$ *about the origin by 45 degrees and the origin is:_____.*

Answer: $\frac{\sqrt{15}}{3}$
Reasoning: The tangent line drawn through the highest point should be parallel to the x-axis. Therefore, after clockwise rotation by 45 degrees, this tangent line should have a slope of 1. By utilizing the slope of the tangent line, we can solve for the coordinates of the point of tangency as $\left(\frac{2\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$ (discarding the solution in the third quadrant). Consequently, we find the distance from the origin to be $\frac{\sqrt{15}}{3}$.

Problem 98. *A convex* n*-gon with interior angles of* n *degrees each, all integers, and all different. The degree measure of the largest interior angle is three times the degree measure of the smallest interior angle. The maximum value that* n *can take is _____.*

Answer: 20
Reasoning: Let the smallest interior angle be m degrees, then $m \leq 59$. The largest interior angle is $3m$ degrees, and the next largest interior angle is at most $3m-1$ degrees, and so on until the second smallest interior angle (the $(n-1)$-th largest) is at most $3m-n+2$ degrees. Therefore, we have:

$$180(n-2) \leq m + 3m + (3m-1) + (3m-2) + \ldots + (3m-n+2)$$

Using $m \leq 59$ for simplification, we get $n^2 + 3n - 482 \leq 0$. Hence, $n \leq 20$.

Problem 99. *In triangle* ABC *with its incenter* I*, if* $3\overrightarrow{IA} + 4\overrightarrow{IB} + 5\overrightarrow{IC} = \overrightarrow{0}$*, then the measure of angle* C *is _____.*

Answer: 90
Reasoning: Extending $I\vec{A}, I\vec{B}$, and $I\vec{C}$ respectively to $3, 4$, and 5 times, I becomes the centroid of the resulting new triangle. Then, we can infer that the ratios of the areas of triangles IAB, IBC, and ICA are $\frac{1}{3\times4} : \frac{1}{4\times5} : \frac{1}{5\times3} = 5 : 3 : 4$. Furthermore, since I is the incenter, we have AB : BC : CA = 5 : 3 : 4. Finally, by the converse of the Pythagorean theorem, angle C is a right angle.

Problem 100. *Given the circle* $x^2+y^2 = 4$ *and the point* P(2, 1)*, two mutually perpendicular lines are drawn through point* P*, intersecting the circle at points* A, B *and* C, D *respectively. Point* A *lies inside the line segment* PB*, and point* D *lies inside the line segment* PC*. The maximum area of quadrilateral* $ABCD$ *is _____.*

Answer: $\sqrt{15}$
Reasoning: Set midpoints of AB, CD to be M, N relatively.

$$S_{ABCD} = \frac{1}{2}(PB \cdot PC - PA \cdot PD) = \frac{1}{2}[(PM + MB)(PN + NC) - (PM- MA)(PN - ND)] = \mathrm{PM} \cdot \mathrm{NC} + \mathrm{PN} \cdot \mathrm{MB}$$

By the Cauchy inequality

$$(\text{PM} \cdot \text{NC} + \text{PN} \cdot \text{MB})^2 \leq \left(PM^2 + PN^2\right)\left(MB^2 + MC^2\right)$$
$$= OP^2\left(2^2 - OM^2 + 2^2 - ON^2\right) = OP^2\left(8 - OP^2\right) = 15$$

Hence, the maximum value of the quadrilateral ABCD area is $\sqrt{15}$.

Problem 101. *Given that the area of triangle* ABC *is 1, and* BC $= 1$*, when the product of the three altitudes of this triangle is maximized,* $\sin A =$

Answer: $\frac{8}{17}$
Reasoning: Since the area of the triangle is fixed, the product of the three altitudes with the three sides is also fixed. To maximize the product of the three altitudes, we should minimize the product of the three sides. Since angle A is fixed with respect to side BC, and the area of the triangle is given by $\frac{1}{2}bc\sin A$, which is constant, we should maximize $\sin A$, so that bc is minimized. As shown in the diagram below, draw line l parallel to BC at a distance of 2 units. Then, draw the perpendicular from B to l, intersecting l at A. Construct the circumcircle O of triangle ABC. In this configuration, A is at a position that maximizes angle A, as the angle subtended at the circumference of the circle is greater than the angle subtended at the center. Calculate AB = AC = $\frac{\sqrt{17}}{2}$ using the Pythagorean theorem, and then use the cosine rule to find the trigonometric value of angle A.

Problem 102. *If each face of a tetrahedron is not an isosceles triangle, then it has at least_____ distinct edge lengths.*

Answer: 3
Reasoning: Each pair of edges needs to be equal.

Problem 103. *In tetrahedron $ABCD$, where $AC = 15$, $BD = 18$, E is the trisect point of AD closer to A, F is the trisect point of BC closer to C, and $EF = 14$. Then, the cosine value of the angle between edges AC and BD is_____.*

Answer: $\frac{1}{2}$
Reasoning: Taking the trisect point G on CD closer to C, we have $EG//AC$ and $FG//BD$, where $EG = 10$ and $FG = 6$.

Then, $\cos\angle EGF = \frac{6^2+10^2-14^2}{2\times6\times10} = -\frac{1}{2}$.

Since the angle between skew lines is acute, the result is $\frac{1}{2}$.

Problem 104. *Given that each face of the tetrahedron has edges of lengths $\sqrt{2}$, $\sqrt{3}$, and 2, the volume of this tetrahedron is _____.*

Answer: $\sqrt{30}_{\overline{12}}$.
Reasoning: Considering a rectangular parallelepiped with three face diagonals of lengths $\sqrt{2}, \sqrt{3}, 2$, the lengths of its three edges are $\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}, \frac{\sqrt{10}}{2}$. Its volume $V' = \frac{\sqrt{30}}{4}$. Therefore, the volume of the tetrahedron is $V = \frac{V'}{3} = \frac{\sqrt{30}}{12}$.

Problem 105. *Given that the line l intersects two parabolas $\Gamma_1 : y^2 = 2px (p > 0)$ and $\Gamma_2 : y^2 = 4px$ at four distinct points $A(x_1, y_1)$, $B(x_2, y_2)$, $D(x_3, y_3)$, and $E(x_4, y_4)$, where $y_4 < y_2 < y_1 < y_3$. Let l intersect the x-axis at point M. Given that $AD = 6BE$, then the value of $\frac{AM}{ME}$ is _____.*

Answer: $\sqrt{3}$
Reasoning: According to the given conditions, the slope of the line l exists. Let $l : y = kx + m$. Then

$$\begin{cases} y^2 = 2px, \\ y = kx + m \end{cases} \Rightarrow ky^2 - 2py + 2pm = 0 \Rightarrow \frac{1}{y_1} + \frac{1}{y_2} = \frac{y_1+y_2}{y_1 y_2} = \frac{1}{m}.$$

Similarly, $\frac{1}{y_3} + \frac{1}{y_4} = \frac{1}{m}$. Hence, $\frac{1}{y_1} + \frac{1}{y_2} = \frac{1}{y_3} + \frac{1}{y_4} \Rightarrow \frac{1}{y_1} - \frac{1}{y_3} = \frac{1}{y_4} - \frac{1}{y_2} \Rightarrow \frac{y_3-y_1}{y_1 y_3} = \frac{y_2-y_4}{y_2 y_4} \Rightarrow \frac{AD}{BE} = \frac{|y_3-y_1|}{|y_2-y_4|} = \frac{|y_1 y_3|}{|y_2 y_4|}$. By $y_1 y_2 = \frac{2pm}{k}$ and $y_3 y_4 = \frac{4pm}{k}$, $\Rightarrow \frac{y_1 y_3}{y_2 y_4} = \frac{y_1 \cdot \frac{4pm}{k y_4}}{\frac{2pm}{k y_1} \cdot y_4} = \frac{2y_1^2}{y_4^2} = 6$. Hence, $\frac{AM}{ME} = \frac{|y_1|}{|y_4|} = \sqrt{3}$.

Problem 106. *Given that $\triangle ABC$ is an acute-angled triangle, with A, B, and C as its internal angles, the minimum value of $2\cot A + 3\cot B + 4\cot C$ is _____.*

Answer: $\sqrt{23}$
Reasoning: By the inequality $(2x - 3y\cos\gamma - 4z\cos\beta)^2 + (3y\sin\gamma - 4z\sin\beta)^2 \geqslant 0$, we can rearrange it to obtain $(2x + 3y + 4z)^2 \geqslant 12(\cos\gamma + 1)xy + 24(1 - \cos(\beta + \gamma))yz + 16(\cos\beta + 1)zx$.

Let's assume the coefficients, and denote $12(\cos\gamma + 1) = 24(1 - \cos(\beta + \gamma)) = 16(\cos\beta + 1) = k$. This leads to the equation $\left(\frac{k}{12} - 1\right)\left(\frac{k}{16} - 1\right) - \sqrt{1 - \left(\frac{k}{12} - 1\right)^2}\sqrt{1 - \left(\frac{k}{16} - 1\right)^2} = 1 - \frac{k}{24} \Rightarrow k = 23$.

Therefore, $(2x+3y+4z)^2 \geqslant 23(xy+yz+zx)$. Also $\cot A \cdot \cot B + \cot B \cdot \cot C + \cot C \cdot \cot A = 1$, thus, $(2\cot A + 3\cot B + 4\cot C)^2 \geqslant 23$. Therefore, the minimum value is $\sqrt{23}$.

Problem 107. *There are 10 points in the plane, with no three points lying on the same line. Using these 10 points as vertices of triangles, such that any two triangles have at most one common vertex, we can form at most _____ triangles.*

Answer: 13
Reasoning: Considering all triangles containing one of the points A, since there is at most one common point (which is A), the other two points of these triangles must be different. Therefore, there are $\left[\frac{10-1}{2}\right] = 4$ triangles containing point A. Similarly, for each point, we can obtain 4 triangles. Hence, there are a total of $4 \times 10 = 40$ triangles. Since each triangle is counted 3 times, at most $\left[\frac{40}{3}\right] = 13$ triangles can be formed. Here is a construction:

	A	B	C	D	E	F	G	H	I	J
First	√	√	√							
Second	√			√	√					
Third	√					√				√
Fourth	√							√	√	
Fifth		√						√		√
Sixth		√			√				√	
Seventh		√		√		√				
Eighth						√	√	√		
Nineth			√		√			√		
Tenth			√	√			√			
Eleventh			√			√			√	
Twelfth				√					√	√
Thirteenth					√		√			√

Problem 108. *Given that quadrilateral $ABCD$ is a parallelogram, with the lengths of AB, AC, AD, and BD being distinct integers, then the minimum perimeter of quadrilateral $ABCD$ is ______.*

Answer: 32
Reasoning: Let AB = a, AD = b, AC = p, BD = q. Obviously, $2\left(a^2+b^2\right)=p^2+q^2$. Without loss of generality, assume a > b and p > q.

Let $r=a^2+b^2$.

If the equation $x^2+y^2=r(\text{x}>\text{y})$ has a unique non-negative integer solution $\begin{cases} x=a \\ y=b \end{cases}$, then p = a + b and q = a − b, and in this case, a parallelogram cannot be formed.

Hence, if the equation $x^2+y^2=r(\text{x}>\text{y})$ has at least 2 sets of non-negative integer solutions, then r is composite, and in its factorization, there are at least 2 primes of the form $4p+1$ multiplied together.

If $r=25=5\times 5$, then a = 4, b = 3, p = q = 5, which contradicts the inequality of AC and BD.

If $r=50=2\times 5\times 5$, then a = b = 5, p = 8, q = 6, which contradicts the inequality of AB and AD.

If $r=65=5\times 13$, then a = 7, b = 4, p = 9, q = 7, which contradicts the inequality of AB and BD.

If $r=85=5\times 17$, then a = 7, b = 6, p = 11, q = 7, which contradicts the inequality of AB and BD.

If $r=130=2\times 5\times 13$, then $a=9$, $b=7$, $p=14$, $q=8$, and in this case, $a+b=16$, so the perimeter is 32.

If $r=170=2\times 5\times 17$, then $a=11$, $b=7$, $p=14$, $q=12$, and in this case, $a+b=18$, so the perimeter is 36.

If $r=221=13\times 17$, then $a=11$, $b=10$, $p=19$, $q=9$, and in this case, $a+b=21$, so the perimeter is 42.

For $r>226$, $a+b>16$, hence the minimum perimeter of $ABCD$ is 32.

Problem 109. *When the two ends of a strip of paper are glued together, forming a loop, it is called a circular ring. Cutting along the bisector of the paper strip will result in two circular rings. When a strip of paper is twisted 180 degrees and then the ends are glued together again, forming a loop, it is called a Mobius strip. Cutting along the bisector of the Mobius strip will result in a longer loop-like structure. If cut along the bisector of this longer loop-like structure, it will yield _____ loop-like structure again.*

Answer: 2
Reasoning:

Examine the topological structure.

As shown in the figure below, cutting along the quadrisection line of the Möbius strip is equivalent to cutting along the $\frac{1}{4}$-line. At this point, the second strip and the third strip are glued together to form a larger loop-like structure, while the first strip and the fourth strip are glued together to form the same loop-like structure. Therefore, there are a total of 2 loop-like structures. (It is also easy to obtain by cutting along the $\frac{1}{n}$-line of the Möbius strip using this method.)

Problem 110. *In the Cartesian coordinate system xOy, let* Γ_1 *be the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1(a>b>0)$ *and* Γ_2 *be the parabola* $y^2=\frac{1}{2}ax$. *They intersect at points A and B, and P is the rightmost point of* Γ_1. *If points O, A, P, and B are concyclic, then the eccentricity of* Γ_1 *is_____.*

Answer: $\frac{\sqrt{6}}{3}$
Reasoning: By symmetry, we know that $\angle OAP=\angle OBP=90^\circ$, so points A and B lie on the circle with OP as its diameter. From

$$\begin{cases}\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\\ x^2-ax+y^2=0\end{cases}$$

$\Rightarrow \frac{c^2}{a^2}x^2-ax+b^2=0$. Then $x_Ax_P=\frac{a^2b^2}{c^2}\Rightarrow x_A=\frac{ab^2}{c^2}\Rightarrow y_A^2=-x_A^2+ax_A=-\frac{a^2b^4}{c^4}+\frac{a^2b^2}{c^2}$. Combining with $y_A^2=\frac{1}{2}ax_A$ gives $-\frac{a^2b^4}{c^4}+\frac{a^2b^2}{c^2}=\frac{1}{2}\cdot\frac{a^2b^2}{c^2}\Rightarrow c^2=2b^2\Rightarrow 3c^2=2a^2$. Therefore, the eccentricity e of ellipse Γ_1 is $e=\frac{c}{a}=\frac{\sqrt{6}}{3}$.

Problem 111. *Given the circle* $\Gamma: x^2+y^2=1$, *points A and B are two points symmetric about the x-axis on the circle. M is any point on the circle* Γ *distinct from A and B. If MA and MB intersect the x-axis at points P and Q respectively, then the product of the abscissas of P and Q is _____.*

Answer: 1
Reasoning: Let $A(m,n),B(m,-n),M\left(x_0,y_0\right)$. Then

$l_{MA}: y-y_0=\frac{y_0-n}{x_0-m}\left(x-x_0\right)$, $l_{MB}: y-y_0=\frac{y_0+n}{x_0-m}\left(x-x_0\right)$. Setting y=0 yield $x_P=x_0-\frac{y_0(x_0-m)}{y_0-n}, x_Q=x_0-\frac{y_0(x_0-m)}{y_0+n}$, So x_Px_Q

$$\begin{aligned}&=x_0^2+\frac{y_0^2(x_0-m)^2-2x_0y_0^2(x_0-m)}{y_0^2-n^2}\\&=x_0^2-\frac{y_0^2\left(x_0^2-m^2\right)}{y_0^2-n^2}\\&=\frac{y_0^2m^2-x_0^2n^2}{y_0^2-n^2}.\end{aligned}$$

Substituting $x_0^2 = 1 - y_0^2$, $m^2 = 1 - n^2$ into the above expression, we get

$$x_P x_Q = \frac{y_0^2\left(1-n^2\right)-\left(1-y_0^2\right)n^2}{y_0^2-n^2} = 1.$$

Problem 112. *If three points are randomly chosen from the vertices of a regular 17-sided polygon, what is the probability that the chosen points form an acute-angled triangle?*

Answer: $\frac{3}{10}$
Reasoning: When selecting any three points, all forming triangles, there are a total of $C_1 7^3 = 680$ triangles, among which there are no right-angled triangles. Classify obtuse-angled triangles based on the length of the longest side. The longest side corresponds to the diagonal of the regular 17-sided polygon, which has 7 different lengths (there are 1, 2, ..., 7 vertices between the two ends), with exactly 17 diagonals of each length. Thus, there are $17(1 + 2 + ... + 7) = 467$ obtuse-angled triangles. Therefore, the probability of forming an acute-angled triangle is $p = \frac{680-476}{680} = \frac{3}{10}$.

Problem 113. *A rook piece moves through each square of a* 2023×2023 *grid paper once, each time moving only one square (i.e., from the current square to an adjacent square). If the squares are numbered from 1 to* n^2 *according to the order in which the rook piece reaches them, let* M *denote the maximum difference in numbers between adjacent squares. Then the minimum possible value of* M *is?*

Answer: 4045
Reasoning: Firstly, it is explained that the minimum possible value of M when operating on a $n \times n$ grid paper is $2n-1$. Firstly, it is explained that M can be equal to $2n-1$. In fact, as long as the rook piece moves in a 'serpentine' manner on the chessboard: moving along the bottom row from the leftmost to the rightmost, then moving up one square, then moving along that row from the rightmost to the leftmost, then moving up one square, and so on. Then it is proved that $M \geq 2n - 1$. By contradiction: Assume $M < 2n - 1$, observe the numbers in the top row. Since the difference between any two adjacent numbers in this row is not greater than $2n-2$, then when the rook piece moves from the smallest number in this row to the largest number, it cannot pass through the squares in the bottom row, because to reach the bottom row, it needs to take at least $n - 1$ steps, and to return, it needs another $n - 1$ steps, and it still needs to spend one step for horizontal movement. This indicates that the rook piece does not pass through the squares in the bottom row when traversing all the numbers in the top row. Similarly, when the rook piece traverses all the numbers in the bottom row, it does not pass through the squares in the top row. This indicates that all the numbers in the top row are either all greater than or all less than all the numbers in the bottom row. Similarly, all the numbers in the leftmost column are either all greater than or all less than all the numbers in the rightmost column. Without loss of generality, assume that all numbers in the leftmost column are greater than those in the rightmost column, and all numbers in the bottom row are greater than those in the top row. Now observe the number A at the top left corner and the number B at the bottom right corner. On one hand, when viewed by column, we have $A > B$, at the bottom right corner. On one hand, when viewed by column, we have A¡B. Contradiction. Therefore, the answer to this problem is $2 \times 2023 - 1 = 4045$.

Problem 114. *Using the 24-hour clock, the probability of the sum of four digits at a certain moment being smaller than the sum of the four digits at 20:21 is _____.*

Answer: $\frac{13}{288}$
Reasoning: Because the sum of the four digits at the time 20:21 is 5, the digits at other times satisfying the condition must sum up to 0, 1, 2, 3, or 4.

When the sum of the four digits is 0, there is 1 possibility.

When the sum is 1, there are 4 possibilities.

When the sum is 2, there are 4 + 6 = 10 possibilities.

When the sum is 3, there are 3 + 4 + 12 = 19 possibilities.

When the sum is 4, there are 1 + 3 + 9 + 6 + 12 = 31 possibilities.

Thus, there are a total of 65 possibilities satisfying the condition.

Therefore, the required probability is $\frac{65}{24\times 60} = \frac{13}{288}$.

Problem 115. *A six-digit number $N = \overline{a_1 a_2 ... a_6}$ composed of non-repeating digits from 1 to 6 satisfies the condition $|a_{k+1} - a_k| \neq 1, (k \in \{1, 2, \cdots, 5\})$. Then the number of such six-digit numbers is _____.*

Answer: 90
Reasoning: According to the problem, we know that a_{k+1} and a_k are not adjacent numbers, there are five cases where the digits are adjacent: (1,2), (2,3), (3,4), (4,5), and (5,6).

(1) If at least one pair is adjacent, there are $5A_2^2 A_5^5 = 1200$ possibilities.

(2) If at least two pairs are adjacent, there are two cases:

(i) Three consecutive numbers are adjacent, resulting in $4 \times 2A_4^4 = 192$ possibilities;

(ii) Two numbers are adjacent, but the two groups of numbers are not adjacent, resulting in $6 \times 2^2 A_4^4 = 576$ possibilities. Therefore, there are a total of 768 possibilities where at least two pairs are adjacent.

(3) If at least three pairs are adjacent, there are three cases:

(i) Four consecutive numbers are adjacent, resulting in $3 \times 2A_3^3 = 36$ possibilities;

(ii) Three consecutive numbers and two adjacent numbers, but the two groups of numbers are not adjacent, resulting in $6 \times 2 \times 2A_3^3 = 144$ possibilities;

(iii) Each of the three pairs has only two adjacent numbers, resulting in $2^3 A_3^3 = 48$ possibilities.

Therefore, there are a total of 228 possibilities where at least three pairs are adjacent.

(4) If at least four pairs are adjacent, there are two cases:

(i) Five consecutive numbers are adjacent, resulting in $2 \times 2A_2^2 = 8$ possibilities;

(ii) Four consecutive numbers and two adjacent numbers, but the two groups of numbers are not adjacent, or both groups consist of three consecutive numbers but are not adjacent, resulting in $3^2 A_2^2 = 24$ possibilities.

Therefore, there are a total of 32 possibilities where at least four pairs are adjacent.

(5) If all five pairs are adjacent, there are 2 possibilities.

By the principle of inclusion-exclusion, we know that the total number of permutations satisfying the condition is $6! - 1200 + 768 - 228 + 32 - 2 = 90$.

Problem 116. *Given $M = \{1, 2, \ldots, 8\}, A, B$ are two distincet subsets of set M, satisfying*

(1) The number of elements in set A is fewer than the number of elements in set B.

(2) The smallest element in set A is larger than the largest element in set B.
Then the total number of ordered pairs (A, B) satisfying these conditions is _____.

Answer: 321
Reasoning: Based on the elements of $A \cup B$, we can discuss:

If $|A \cup B| = 3$, then there is only one way to split these three elements, giving $C_8^3 - 1 = 56$ possibilities.

If $|A \cup B| = 4$, then there is only one way to split these four elements, giving $C_8^4 = 70$ possibilities.

If $|A \cup B| = 5$, then there are two ways to split these five elements, giving $C_8^5 \times 2 = 112$ possibilities.

If $|A \cup B| = 6$, then there are two ways to split these six elements, giving $C_8^6 \times 2 = 56$ possibilities.

If $|A \cup B| = 7$, then there are three ways to split these seven elements, giving $C_8^7 \times 3 = 24$ possibilities.

If $|A \cup B| = 8$, then there are three ways to split these eight elements, giving $C_8^8 \times 3 = 3$ possibilities.

In summary, the number of ordered pairs satisfying the conditions is $56 + 70 + 112 + 56 + 24 + 3 = 321$.

Problem 117. *Using six different colors to color each edge of the regular tetrahedron $ABCD$, each edge can only be colored with one color and edges sharing a vertex cannot have the same color. The probability that all edges have different colors is _____.*

Answer: $\frac{3}{17}$
Reasoning: Classify according to whether the opposite edges of the tetrahedron have the same color.

If all three pairs of opposite edges have the same color, meaning three colors are used, then there are A_6^3 different coloring schemes.

If only two pairs of opposite edges have the same color, meaning four colors are used, then there are $C_3^2 A_6^4$ different coloring schemes.

If only one pair of opposite edges have the same color, meaning five colors are used, then there are $C_3^1 A_6^5$ different coloring schemes.

If all edges have different colors, meaning six colors are used, then there are A_6^6 different coloring schemes.

Therefore, the probability that all edges have different colors is $\frac{A_6^6}{A_6^3 + C_3^2 A_6^4 + C_3^1 A_6^5 + A_6^6} = \frac{3}{17}$.

Problem 118. *The king summons two wizards into the palace. He demands Wizard A to write down 100 positive real numbers on cards (allowing duplicates) without revealing them to Wizard B. Then, Wizard B must accurately write down all of these 100 positive real numbers. Otherwise, both wizards will be beheaded. Wizard A is allowed to provide a sequence of numbers to Wizard B, where each number is either one of the 100 positive real numbers or a sum of some of them. However, he cannot tell Wizard B which are the numbers on the cards and which are the sums of numbers on the cards. Ultimately, the king decides to pull off the same number of beards from both wizards based on the count of these numbers. Without the ability to communicate beforehand, the question is: How many beard pulls does each wizard need to endure at least to ensure their own survival?*

Answer: 101

Reasoning: If only 100 hint numbers are given, it's impossible to distinguish whether all 100 numbers are on the cards or there are 99 numbers on the cards, and the largest hint number is the sum of the number on the 100th card and another number on the card. Therefore, at least 101 hint numbers are needed. For the 101 hint numbers, we can write down powers of 2 from 2^1 to 2^{100} on the cards, and give hints for these 100 numbers as well as their sum. In this way, by using the number 2, we can determine that there must be a number on one of the cards that is not greater than 2. Then, using the number 4, we can determine that there must be another number on one of the cards that is not greater than 4, and so on. This process allows us to sequentially determine that the numbers on the 100 cards are not greater than 2 raised to the power of 1 to 100, and then we can infer the specific values of these 100 numbers based on their sum.

Problem 119. *Using 1×1, 2×2, and 3×3 tiles to cover a 23×23 floor (without overlapping or leaving gaps), what is the minimum number of 1×1 tiles needed? (Assuming each tile cannot be divided into smaller tiles).*

Answer: 1

Reasoning: Paint the $3k+1$ columns black, where $k = 0, 1, 2, \ldots, 7$, and the rest white. Then, there are an odd number of white squares, but both 2×2 and 3×3 cover an even number of white squares. Therefore, at least one 1×1 tile is needed. Use 2×2 and 3×3 tiles to first form 2×6 and 3×6 sections, which can then be combined to form an 11×12 grid with 4 copies of 11×12 and 1 1×1 tile, completing the construction of a 23×23 grid.

Problem 120. *In a number-guessing game, the host has prearranged a permutation of the numbers 1 to 100, and participants also need to provide a permutation of these 100 numbers. Interestingly, as long as the permutation provided by a participant has at least one number whose position matches that of the host's permutation, it is considered a successful guess. How many participants are needed to ensure that at least one person guesses correctly?*

Answer: 51

Reasoning: The first person guesses that the first 51 numbers are from 1 to 51, the second person guesses that the first 51 numbers are $2, 3, 4, \ldots, 51, 1$, and so on in a rotating fashion. If all 51 people fail to guess correctly, it implies that the numbers from 1 to 51 are all in the last 49 positions, which is clearly contradictory. For 50 people, the permutation arranged by the host can be constructed as follows: fill in numbers from 1 to 100 in the positions 1 to 100 successively, ensuring that each number differs from the guesses of the 50 participants (at least 50 different numbers). If, for any position k, it's not possible to do so, it implies that at least 50 numbers that could have been placed in position k have already been placed elsewhere. Since any number not yet placed can be placed in at most 50 different positions (including position k), there will always be a number that can replace the one initially placed at position k. This process ensures that all 50 participants fail to guess correctly.

Problem 121. *There is a stack of 52 face-down playing cards on the table. Mim takes 7 cards from the top of this stack, flips them over, and puts them back at the bottom, calling it one operation. The question is: how many operations are needed at least to make all the playing cards face-down again?*

Answer: 112

Reasoning: Starting from the top of this stack of playing cards, color the 52 cards as follows: the top three blue, the next four red, the next three blue, the next four red, and so on until all cards are colored. Note that 52 mod 7 equals 3, meaning that each of Mim's operations does not change the distribution of colors in this stack of playing cards. If we only consider the blue cards, there are a total of eight groups of blue cards (each group consisting of three cards). After one operation, the top group of blue cards is flipped over and moved to the bottom. Therefore, flipping all the blue cards requires 8 operations, and flipping them back requires a total of 16 operations. Similarly, there are seven groups of red cards, and flipping all the red cards requires 14 operations. Thus, it takes $[14, 16] = 112$ operations to make all the playing cards face-down again.

Problem 122. *There are two segments of length* $3n(0 \leq n \leq 1011)$. *How many different shapes of triangles can be formed from these 2024 segments? (Congruent triangles are considered the same.)*

Answer: 511566

Reasoning: From the triangle inequality, it's known that the triangle must be isosceles, and the length of the legs is not shorter than the length of the base. Then, classifying by the length of the legs, there are $1 + 2 + \ldots + 1011 = 511566$ types of triangles.

Problem 123. *Let* A *and* B *be two subsets of the set* $\{1, 2, \ldots, 20\}$, *where* $A \cap B = \varnothing$, *and if* $n \in A$, *then* $2n + 2 \in B$. *Let* $M(A)$ *denote the sum of the elements of* A. *The maximum value of* $M(A)$ *is:*

Answer: 39

Reasoning: From 2n+2 $\leq 20 \Rightarrow n \leq 9$.

According to the pigeonhole principle, A can have at most 6 elements. When $A = \{9, 8, 7, 6, 5, 4\}$, we get the maximum value of $M(A)$ as 39.

Problem 124. *Alice and Bob are playing a game. They write down four expressions on four cards:* $x + y$, $x - y$, $x^2 + xy + y^2$, *and* $x^2 - xy + y^2$. *They place these four cards face down on the table, then randomly choose one card to reveal its expression. Alice can pick two of the four cards and hand the other two to Bob, then all four cards are revealed. Alice can assign a value (real number) to one of the variables* x *or* y *and inform Bob of which variable she has assigned and what value. Afterwards, Bob assigns a value (real number) to the other variable. Finally, they each calculate the product of the values on their two cards, and the person with the larger product wins. Who has a winning strategy? A. Alice B. Bob C. Neither of them has a winning strategy.*

Answer: A

Reasoning: Alice has a winning strategy.

Firstly, let A and B represent the products of the two cards in Alice's and Bob's hands respectively.

If $x - y$ or $x + y$ is revealed, then Alice chooses any two hidden cards. Otherwise, she picks one hidden card and one revealed card, ensuring she doesn't get both $x - y$ and $x + y$.

If Bob gets both $x - y$ and $x + y$, then Alice can choose $y = 1$. In this case,

$A = (x^2 - xy + y^2)(x^2 + xy + y^2) = x^4 + x^2 + 1$,
$B = (x - y)(x + y) = x^2 - y^2 = x^2 - 1$,
thus $A - B = x^2 + 2 > 0$, and Alice wins.
If $B = (x - y)(x^2 + xy + y^2) = x^3 - y^3$,
$A = (x + y)(x^2 - xy + y^2) = x^3 + y^3$,
then Alice chooses $y > 0$.
If $A = x^3 - y^3, B = x^3 + y^3$, then Alice chooses $y < 0$.
If $A = (x - y)(x^2 - xy + y^2)$,
$B = (x + y)(x^2 + xy + y^2)$,
then $A - B = -4x^2y - 2y^3 = -2y(y^2 + 2x^2)$, in this case Alice chooses $y < 0$.
If $A = (x + y)(x^2 + xy + y^2)$,
$B = (x - y)(x^2 - xy + y^2)$, then $A - B = 4x^2y + 2y^3 = 2y(y^2 + 2x^2)$, and Alice chooses $y > 0$.

In conclusion, Alice has a winning strategy.

Problem 125. *Find the smallest integer $k > 2$ such that any partition of $\{2, 3, \ldots, k\}$ into two sets must contain at least one set containing a, b, and c (which are allowed to be the same), satisfying $ab = c$.*

Answer: 32
Reasoning: Firstly, we provide a counterexample for $k = 31$.

Let $A = \{2, 3, 16, 17, \ldots, 31\}$ and $B = \{4, 5, \ldots, 15\}$.

If $k < 31$, then simply remove integers greater than k from sets A and B respectively.

Next, we prove that the conclusion holds when $k = 32$.

Conversely, if the conclusion does not hold, then 2 and 4, and 4 and 16 are not in the same set respectively. Thus, 2 and 16 are in the same set. Similarly, 4 and 8 are in the same set. In this case, 32 cannot be in any set ($32 = 2 \times 16 = 4 \times 8$).

Therefore, the minimum value of k is 32.

Problem 126. *On a plane, there are 2019 points. Drawing circles passing through these points, a drawing method is called "k-good" if and only if drawing k circles divides the plane into several closed shapes, and there is no closed shape containing two points. Then, find the minimum value of k , it ensures that no matter how these 2019 points are arranged, there always exists a k-good drawing method.*

Answer: 1010
Reasoning: First, we prove that $k \geqslant 1010$.

Take 2019 points on the same line. Obviously, among these 2019 points on the line, at least 2018 of them need to be passed through by a circle.

Furthermore, since each circle can pass through the points on the line at most twice, if $k \leqslant 1009$, then the number of points passed through by these k circles does not exceed 2018. Note that when equality holds, the first and last points are not contained within any circle, leading to a contradiction.

Next, we strengthen the proof that no matter how these 2019 points are arranged, there always exists a 1010-good drawing method.

The strengthened proposition is: "Prove: there exists a 1010-good drawing method such that these 1010 circles pass through the same point."

Perform an inversion transformation with a point on the plane not coinciding with any of the 2019 points and not lying on the circumcircle of any three points as the center of inversion. Then, after inversion, to prove the strengthened proposition, it suffices to prove that for any 2019 non-collinear points on the plane, there exist 1010 lines such that each pair of points has at least one line passing through them.

A line is said to "bisect" a set of points if the difference between the number of points on each side of the line is either 1 or equal.

Lemma:

For any two sets of points A and B, there exists a line l that bisects A and B.

Proof: The portion of a plane divided by a line is called a half-plane. Obviously, there exists a half-plane containing half of the points in both A and B, i.e., $2\times$ the number of points in A in the half-plane $-|A| \leqslant 1$.

It is evident that there exists a half-plane containing fewer than half of the elements in B and half of the elements in A; there exists a half-plane containing more than half of the elements in B and half of the elements in A; there exists a way to rotate and translate the half-plane so that it can be transformed from any half-plane containing half of the elements in A to another, and the route of translation and rotation through half-planes. Therefore, by the intermediate value theorem, the proposition is proved.

A region including a given point is called a figure formed by closed lines, and the modulus of a region is the number of points in the region. Start by connecting a sufficiently large polygon enclosing all the points. Obviously, there is only one region at the beginning.

Consider drawing 1010 lines in order. Consider making each line bisect the two regions with the largest moduli at the moment, defining this as one operation. Next, we prove that this operation can be performed 1010 times.

The first operation generates a new region, and thereafter each operation generates two new regions. By induction, it is easy to see that the modulus of the region with the maximum modulus is no more than twice the modulus of the region with the minimum modulus, so if there is a region with a modulus of 1, then the moduli of the other regions must be either 1 or 2.

Therefore, when the last operation stops, there is at most one region with a modulus of 2 left. Due to parity considerations, there are 2018 regions at this point, and 1009 lines have been drawn. Drawing one more line to separate the two points left in the same region is enough. Thus, it is proved that it is possible to sequentially draw 1010 closed shapes such that no closed shape contains two points. Thus, the proposition is proved.

Problem 127. *Zheng flips an unfair coin 5 times. If the probability of getting exactly 1 head is equal to the probability of getting exactly 2 heads and is nonzero, then the probability of getting exactly 3 heads is ____.*

Answer: $\frac{40}{243}$.
Reasoning: Let the probability of getting a head be p.

According to the given conditions, we have $C_5^1 p(1-p)^4 = C_5^2 p^2(1-p)^3$.

Solving this equation, we find $p = \frac{1}{3}$.

Therefore, the probability of getting exactly 3 heads is $C_5^3 p^3(1-p)^2 = \frac{40}{243}$.

Problem 128. *Let $a_1, a_2, \ldots, a_6$ be any permutation of $\{1, 2, \ldots, 6\}$. If the sum of any three consecutive numbers cannot be divided by 3, then the number of such permutations is ____.*

Answer: 96
Reasoning: Taking the numbers in the set modulo 3, we get two 0s, two 1s, and two 2s. Thus, permutations satisfying the condition that the sum of any three consecutive numbers cannot be divided by 3 are 002211, 001122, 112200, 110022, 220011, 221100, 011220, 022110, 100221, 122001, 211002, and 200112.

Therefore, the number of permutations satisfying the condition is $12 \times A_2^2 \times A_2^2 \times A_2^2 = 96$.

Problem 129. *Given an integer $n > 2$. Now, there are n people playing a game of "Passing Numbers". It is known that some people are friends (friendship is mutual), and each person has at least one friend. The rules of the game are as follows: each person first writes down a positive real number, and the n positive real numbers written by everyone are all different; then, for each person, if he has k friends, he divides the number he wrote by k, and tells all his friends the result obtained; finally, each person writes down the sum of all the numbers he hears. The question is: what is the minimum number of times that someone writes down different numbers?*

Answer: 2
Reasoning: At least 2 people write down different numbers twice.

Let the n people be denoted as $A_1, A_2, \ldots, A_n$.

Consider the following friendship relationship: A_i is a friend of A_j if and only if $|i - j| = 1 (1 \leq i, j \leq n)$.

Under this friendship relationship, A_1 writes down the number 1, A_i writes down the number $i + 1 (2 \leq i \leq n - 1)$, and A_n writes down the number $\frac{n+1}{2}$. Then after one pass,

the numbers written down by $A_2, A_3, \ldots, A_{n-1}$ remain unchanged. However, the number written down by A_1 changes from 1 to $\frac{3}{2}$, and the number written down by A_n changes from $\frac{n+1}{2}$ to $\frac{n}{2}$.

Hence, at this point, there are 2 people who write down different numbers twice.

Next, we prove that: at least 1 person writes down a smaller number the second time.

For each person, define their value as the ratio of the number they wrote down the first time to the number of their friends. Let the value of the person with the highest value among all be M.

Suppose person B has a value of M, and B has k friends. Then the number written down by B the first time is kM, and according to the rules of the game, the number written down by B the second time is at most kM. If the number written down by B the second time is smaller than kM, then the conclusion holds; if the number written down by B the second time is kM, then the value of each of his friends is also M.

Construct a graph G, where the vertices are the n people, and there is an edge between two vertices if and only if the two people are friends. Consider the connected component G' where B is located.

From the above discussion, if the conclusion does not hold, then every person represented by a vertex in G' has a value of M. Since the degree of each vertex in G' is at least 1 and at most $|V| - 1$ (where $|V|$ is the number of vertices in G'), by the pigeonhole principle, there must be two vertices with the same degree, which means the two people represented by these vertices wrote down the same number the first time, contradicting the condition. Thus, the conclusion holds.

Note that the sum of the numbers written down by all people the first time is equal to the sum of the numbers written down by all people the second time. From the conclusion, we know that at least one person writes down a smaller number the second time, thus, there must be another person who writes down a larger number the second time, so at least 2 people write down different numbers twice.

Problem 130. *A restaurant can offer 9 types of appetizers, 9 types of main courses, 9 types of desserts, and 9 types of wines. A company is having a dinner party at this restaurant, and each guest can choose one appetizer, one main course, one dessert, and one type of wine. It is known that any two people's choices of the four dishes are not completely identical, and it is also impossible to find four people on the spot who have three identical choices but differ pairwise in the fourth choice (for example, there are no 9 people who have the same appetizer, main course, and dessert, but differ pairwise in the wine). Then, at most how many guests can there be?*

Answer: 5832
Reasoning: Consider the general case, replacing 9 with n.

Consider unordered triples (x, y, z), where x, y, z belong to three different categories of dishes. There are $\binom{4}{3} \times n \times n \times n = 4n^3$ such triples.

Because no n guests can simultaneously choose three identical dishes, for any triple (x, y, z), it can belong to at most $n-1$ guests.

Moreover, each guest chooses one dish from each of the four categories, so each guest has 4 triples.

Let there be x guests.

Then $4x \leq 4n^3(n-1) \Rightarrow x \leq n^3(n-1)$.

Next, we prove that when there are $n^3(n-1)$ guests, the conditions of the problem can be satisfied.

We consider four categories of dishes as four sets A, B, C, D, and each set contains n dishes labeled as $1, 2, \ldots, n$.

We will remove the following selections: the sum of the four numbers chosen is exactly divisible by n. There are n^3 such selections to remove. The remaining number of selections is $n^4 - n^3$, which corresponds to $n^4 - n^3$ guests, each with one selection.

According to the selection rules, we know that no two people choose the same four dishes. (1)

If a particular triple appears more than $n-1$ times, select n guests containing this triple. According to conclusion (1), there exists a category of dishes such that the n dishes chosen by these guests from this category are all different.

Because each category of dishes has only n dishes, these n guests cover all dishes in this category.

Since $1, 2, \ldots, n$ form a complete residue system modulo n, there must be an integer $i(1 \leq i \leq n)$ such that $n \equiv x + y + z + i \pmod{n}$.

However, such selections have been removed, which is a contradiction.

Therefore, the selections of these $n^4 - n^3$ dishes satisfy the conditions of the problem.

Thus, the maximum possible number of guests is $n^4 - n^3$.

In this case, when $n = 9$, the maximum number of guests is $9^4 - 9^3 = 5832$.

Problem 131. *Alice and Bob are playing hide-and-seek. Initially, Bob selects a point B inside a unit square (without informing Alice). Then, Alice sequentially selects points $P_0, P_1, \ldots, P_n$ on the plane. After each selection of a point $P_k (1 \leq k \leq n$, and at this point, Alice has not yet chosen the next point), Bob informs Alice which of the points P_k and P_{k-1} is closer to point B. After Alice selects P_n and receives Bob's response, she chooses a final point A. If the distance between A and B does not exceed $\frac{1}{2020}$, Alice wins. Otherwise, Bob wins. When $n = 18$, which of the following options is correct? A. Alice cannot guarantee victory. B. Alice can guarantee victory.*

Answer: A

Reasoning: It suffices to prove that even under optimal conditions for each step, Alice still cannot guarantee victory.

Note that if we draw the perpendicular bisector of $P_k P_{k-1}$ for each $k (1 \leq k \leq n)$, Alice can determine on which side of the perpendicular bisector point B lies. Consequently, starting from the selection of P_1, after each point is chosen, Alice can narrow down the range where point B is guaranteed to be by at most $\frac{1}{2}$. Thus, after selecting n points, Alice can ensure that point B lies within an area of no less than $\frac{1}{2^n}$.

If Alice can guarantee victory, she should be able to cover an area of no less than $\frac{1}{2^n}$ with a disk of radius $\frac{1}{2020}$.

Therefore, we must have $\pi\left(\frac{1}{2020^2}\right) < 4 \times \frac{1}{2^{20}} = \frac{1}{2^{18}} = \frac{1}{2^n}$ when $n = 18$.

Thus, inequality (1) does not hold.

Hence, Alice cannot guarantee victory.

Problem 132. *Anna, Carl take turns selecting numbers from the set $\{1, 2, \cdots, p-1\}$ (where p is a prime greater than 3). Anna goes first, and each number can only be selected once. Each number chosen by Anna is multiplied by the number Carl selects next. Carl wins if, after any round, the sum of all products computed so far is divisible by p. Anna wins if, after all numbers are chosen, Carl has not won. Which of the following options is correct?*

A. Anna has a winning strategy. B. Carl has a winning strategy. C. Both players have no winning strategy.

Answer: B

Reasoning: Carl has a winning strategy.

Carl's winning strategy is to choose the number $p - a$ whenever Anna selects the number a in a round.

Next, we prove that Carl's strategy guarantees his victory.

Conversely, if Carl's selection strategy does not lead to victory, specifically, if he still has not won after all numbers are chosen, then it implies that p does not divide $1(p-1) + 2(p-2) + \cdots + \frac{p-1}{2}\left(p - \frac{p-1}{2}\right)$.

Let S denote:

$$S = 1(p-1) + 2(p-2) + \cdots + \frac{p-1}{2}\left(p - \frac{p-1}{2}\right)$$

$$= p\left(1+2+\cdots+\frac{p-1}{2}\right)-\left(1^2+2^2+\cdots+\left(\frac{p-1}{2}\right)^2\right)$$

$$= p\left(1+2+\cdots+\frac{p-1}{2}\right)-\frac{1}{6}\cdot\frac{p-1}{2}\cdot\frac{p+1}{2}p$$

Thus, p does not divide S if and only if $24 \mid (p^2-1)\cdot(2)$.

Since p is an odd prime number,

$$8 \mid (p^2-1) \Rightarrow 8 \mid (p-1)(p+1)$$

Combining equations (1) and (2), we have

$$3 \nmid (p^2-1) \Rightarrow 3 \nmid (p-1)(p+1)$$

Thus, $3 \mid p$, which means $p=3$, contradicting the conditions.

Therefore, Carl's strategy guarantees his victory.

Problem 133. *Xiao Ming is playing a coin game with three doors. Each time he opens a door, it costs him 2 coins. After opening the first door, he can see the second door. Upon opening the second door, two equally likely options appear: either return to the outside of the first door or proceed to the third door. Upon opening the third door, three equally likely options appear: either return to the outside of the first door, stay in place and need to reopen the third door, or pass the game. If Xiao Ming wants to pass the game, on average, he needs to spend how many coins?*

Answer: 22

Reasoning: List all possible paths:

(1) First door → Second door → First door, using 4 coins, with a probability of $\frac{1}{2}$;

(2) First door → Second door → Third door → First door, using 6 coins, with a probability of $\frac{1}{6}$;

(3) First door → Second door → Third door → Third door, using 6 coins, with a probability of $\frac{1}{6}$;

(4) First door → Second door → Third door → Pass, using 6 coins, with a probability of $\frac{1}{6}$.

Let E_1 be the average number of coins needed to pass from the first door, and E_2 be the average number of coins needed to pass from the third door. According to the problem:

$E_1 = \frac{1}{2}(4+E_1)+\frac{1}{2}(4+E_2)$

$E_2 = \frac{1}{3}(2+E_1)+\frac{1}{3}(2+E_2)+\frac{1}{3}\times 2.$

Solving yields $E_1 = 22, E_2 = 14$.

Thus, on average, it takes 22 coins to pass from the first door.

Problem 134. *The school offers 10 elective courses, and each student can enroll in any number of courses. The director selects k students, where although each student's combination of courses is different, any two students have at least one course in common. At this point, it is found that any student outside these k students cannot be classmates with these k students regardless of how they enroll (having one course in common is enough to be classmates). Then $k =$ _____.*

Answer: 512

Reasoning: Let S be a set with ten elements. According to the problem, $A_1, A_2, \cdots, A_k$ are subsets of S, each pairwise intersecting non-empty and mutually distinct. Any other subset of S cannot intersect all of $A_1, A_2, \cdots, A_k$.

First, note that there are 2^{10} subsets of S, and they can be paired up to form 2^9 pairs of complements. Thus, $k \leq 2^9$.

Secondly, if $k < 2^9$, then besides $A_1, A_2, \cdots, A_k$, all the other subsets must contain a pair of complementary subsets, denoted as C and D. Hence, there also exist $A_i \cap C = \varnothing$ and $A_j \cap D = \varnothing$.

Since C and D are complementary, it follows that $A_i \subset D$ and $A_j \subset C$. Thus, $A_i \cap A_j = \varnothing$, which is a contradiction.

Therefore, $k = 2^9$ or 512.

Problem 135. *Let n be a positive integer. Now, a frog starts jumping from the origin of the number line and makes $2^n - 1$ jumps. The process satisfies the following conditions:*

(1) The frog will jump to each point in the set $\{1, 2, 3, \cdots, 2^n - 1\}$ exactly once, without missing any.

(2) Each time the frog jumps, it can choose a step length from the set $\{2^0, 2^1, 2^2, \cdots\}$, and it can jump either left or right.

Let T be the reciprocal sum of the step lengths of the frog. When $n = 2024$, the minimum value of T is _____.

Answer: 2024

Reasoning: For a positive integer n, we prove that the minimum value of T is n.

First, we provide an estimation.

Initially, we notice that the frog's jump length must be one of the terms in the set $\{2^0, 2^1, \cdots, 2^{n-1}\}$; otherwise, it would jump out of bounds.

Let a_i $(0 \leq i \leq n-1)$ denote the number of times the frog jumps 2^i steps. According to the conditions, we have $a_0 + a_1 + \cdots + a_{n-1} = 2^n - 1$.

Lemma: For any $k = 1, 2, \cdots, n$, we have $a_{n-1} + \cdots + a_{n-k} \leq 2^n - 2^{n-k}$.

To prove this, let $m = n - k$ and consider the jump lengths modulo 2^m. We categorize jumps less than 2^{m-1} as "small jumps" and those greater than or equal to 2^m as "big jumps". It is notable that small jumps change the residue class modulo 2^m, while big jumps do not.

For each residue class modulo 2^m, the frog can make at most $\frac{2^n}{2^m} - 1$ big jumps. Therefore, the number of big jumps is at most $2^m \left(\frac{2^n}{2^m} - 1\right) = 2^n - 2^m$.

The lemma is proved.

For example, when $n = 3$: for $k = 1$, $a_2 \leq 2^3 - 2^2$ implies that there can be at most four jumps of length 4; for $k = 2$, $a_2 + a_1 \leq 2^3 - 2^1$ implies that there can be at most six jumps of length 2 or 4; for $k = 3$, $a_2 + a_1 + a_0 \leq 2^3 - 2^0$ implies a total of at most 7 jumps. To achieve the maximum S, it is observed that when $a_2 = 4, a_1 = 2, a_0 = 1$, the maximum is attained. Therefore, it is speculated that the maximum value is attained when $a_m = 2^m$.

Let $A = a_0 + a_1 + \cdots + a_{n-1} = 2^n - 1$, and $T = a_0 + \frac{a_1}{2} + \cdots + \frac{a_{n-1}}{2^{n-1}}$. Then, according to the lemma, we have:

$A - T \leq 2^n - n - 1$

$\Rightarrow T \geq n$

Equality holds when $a^m = 2^m$.

Next, we use induction to construct two types of paths such that the frog jumps to $\{0, 1, \cdots, 2^n - 1\}$ exactly once, stopping at x where $x \in \{1, 2n - 1\}$.

When $n = 2$, there are two paths: $\{0, 2, 1, 3\}$ and $\{0, 2, 3, 1\}$.

Assume the claim holds for n, we prove it for $n + 1$.

(i) By the induction hypothesis, there is a path from 0 to 2: $\{0, 2, 4, \cdots, 2^{n+1} - 2\}$. (ii) Similarly, there is a path from 1 to $2^{n+1} - 1$: $\{1, 3, 5, \cdots, 2^{n+1} - 1\}$. (iii) Connecting these two paths requires one step from 2 to 1.

Therefore, we can use the path $0 \to (2^{n+1} - 2) \to (2^{n+1} - 1) \to 1$.

Hence, the minimum value of T is n.

In this question, when $n = 2024$, the answer is 2024.

Problem 136. *Given that there are 66 dwarves with a total of 111 hats, each hat belonging to a specific dwarf, and each hat is dyed in one of 66 colors. During the holiday, each dwarf wears his own hat. It is known that for any holiday, the colors of the hats worn by all dwarves are different. For any two holidays, there is at least one dwarf who wears hats of different colors on the two occasions. The question is: how many holidays can the dwarves celebrate at most?*

Answer: 2^{22}

Reasoning: The problem examines the maximum number of different perfect matches for a bipartite graph with two sets of the same number of vertices under the condition of a specified number of edges, and clarifies the basic structure of the graph when obtaining the maximum value.

The problem structure is described using a graph.

Let $V = \{v_1, v_2, \cdots, v_{66}\}$ be the set of 66 dwarves, and $U = \{u_1, u_2, \cdots, u_{66}\}$ be the set of 66 colors.

If dwarf v_i has a hat of color u_j, then draw an edge between v_i and u_j, and let E be the set of all such edges, obtaining the graph $G(V, U; E)$. Each way of wearing hats that satisfies the requirements for every holiday is a perfect match of G. The problem is to find the maximum number $f(G)$ of different perfect matches for graph G.

The lemma states that for a bipartite graph $G\left(V_1, V_2; E\right)$ with $|V_1| = |V_2|$, it holds that $f(G) \leq 2^{\left[\frac{|E|-|E_1|}{2}\right]}$, where $[x]$ represents the largest integer not exceeding the real number x.

Proof is by mathematical induction.

It is easy to prove that when $|V_1| = |V_2| = 1$, the conclusions hold true.

Suppose the conclusions hold when $|V_1| = |V_2| = n - 1 (n \geq 2)$. Consider the case when $|V_1| = |V_2| = n$.

Consider the bipartite graph $G\left(V_1, V_2; E\right)$.

If G has isolated vertices, then $f(G) = 0$, and the conclusions hold true. Otherwise, let $a \in V_1$ be the vertex in G with the smallest degree $N (N \geq 1)$, and let $b_1, b_2, \cdots, b_N$ be the neighbors of a. Let $G_i = (U_i, U_i'; E_i)$ denote the bipartite graph obtained by deleting vertex a and vertices b_i, as well as the edges incident to a or b_i.

Then $|U_i| = |U_i'| = n - 1$, and $|E_i| \leq |E| - (2N - 1)$.

By the induction hypothesis, it follows that $f\left(G_i\right) \leq 2^{\left[\frac{|E_i|-|U_i|}{2}\right]} \leq 2^{\left[\frac{|E|-(2N-1)-(n-1)}{2}\right]} = 2^{\left[\frac{|E|-n}{2}\right]} \frac{1}{2^{N-1}}$.

Therefore, $f(G)=f\left(G_1\right)+f\left(G_2\right)+\cdots+f\left(G_N\right) \leq \frac{N}{2^{N-1}} 2^{\left[\frac{|E|-n}{2}\right]}$.

By Bernoulli's inequality, $2^{N-1} \geq 1+(N-1)=N$. Thus, $f(G) \leq \frac{N}{2^N} 2^{\left[\frac{|E|-n}{2}\right]} \leq 2^{\left[\frac{|E|-n}{2}\right]}$.

The lemma is proved.

In particular, when $n=66,|E| \leq 111$,

$f(G) \leq 2^{\left[\frac{111-66}{2}\right]}=2^{22}$.

On the other hand, let graph G be composed of 22 complete bipartite graphs $K_{2,2}$ and one bipartite graph $K_{1,1}$. In this case, $|E|=110$, and each complete bipartite graph $K_{2,2}$ has 2 perfect matches, so $f(G)=2^{22}$.

Therefore, the maximum number of different perfect matches is 2^{22}.

Problem 137. *There are n cards, each labeled with the numbers $1,2,\cdots,n$. These n cards are distributed among 17 people, with each person receiving at least 1 card. Then, there is always one person who receives cards with numbers x and y, where $x>y$, and $118x \leqslant 119y$. The smallest positive integer n that satisfies this condition is _____.*

Answer: 2023

Reasoning: Consider the residue classes modulo 17, where the residue class with residue i is represented as $17k+i$ (k is a natural number $1 \leqslant i \leqslant 17$). Let A_i denote the residue class with residue i. For any x and y in each A_i, we have $x-y \geqslant 17$. Then

$$\begin{aligned} & y \leqslant x-17, x \leqslant n \\ & \Rightarrow 119y-118x \leqslant 119(x-17)-118x \\ & \quad = x-2023 \leqslant n-2023. \end{aligned}$$

If $n<2023$, then $11y-118x \leqslant x-2023<0 \Rightarrow 119y<118x$, and $118x \leqslant 119y$, contradiction. Therefore, when $n<2023$, the condition is not satisfied.

When $n=2023$, since $2023=17\times119$, it is known that $118\times17+i(i=0,1,\cdots,17)$ are 18 numbers not exceeding n. By the Pigeonhole Principle, among these 18 numbers $\times17+i(i=0,1,\cdots,17)$, there must be two numbers x and $(x>y)$ in the possession of the same person.

Let $x=118\times17+x_1$ and $y=118\times17+y_1$, where $17 \geqslant x_1 \geqslant y_1 \geqslant 0$. then

$$\begin{aligned} & 118x-119y \\ & =118\left(118\times17+x_1\right)-119\left(118\times17+y_1\right) \\ & =\left(118x_1-119y_1\right)-118\times17 \\ & \leqslant 118x_1-118\times17 \\ & \leqslant 118\times17-118\times17=0. \end{aligned}$$

Therefore, when n=2023, the condition is satisfied.

In conclusion, the smallest positive integer n satisfying the condition is 2023.

Problem 138. *let $a_1,a_2,\cdots,a_9$ be a permutation of $1,2,\cdots,9$. If the permutation $C=(a_1,a_2,\cdots,a_9)$ can be obtained from $1,2,\cdots,9$, by swapping two elements at most 4 times, the total number of permutations satisfying this condition is _____.*

Answer: 27568

Reasoning: Since there are 9 single cycles in the permutation $(1,2,\cdots,9)$, if the permutation $C=(a_1,a_2,\cdots,a_9)$ can be obtained from the permutation $(1,2,\cdots,9)$ by swapping at most 4 times, it is known from analysis that the number of cycles in C must be no less than 5. Below is the classification counting:

(1) 9 cycles, there is 1.

(2) 8 cycles, then 1 2-cycle, there are $C_9^2 \cdot (2-1)! = 36$ permutations.

(3) 7 cycles, as $3+6\times 1 = 2+2+5\times 1$, there are $C_9^3 \cdot (3-1)! + \frac{C_9^2 C_7^2}{2!} - 546$ permutations.

(4) 6 cycles, as $4+5\times1=3+2+4\times1=2+2+2+3\times1$ there are $C_9^4{\cdot}3!{+}C_9^3C_7^2{\cdot}2!{+}\frac{C_9^2C_7^2C_5^2}{3!} = 4536$ permutations.

(5) 5 circles, as $5+4\times 1 = 4+2+3\times 1 = 3+3+3\times 1 = 3+2+2+2\times 1 = 2+2+2+2+2+1$, $C_9^5 \cdot 4! + C_9^4 C_5^2 \cdot 3! + C_9^3 C_6^3 \cdot 2! + \frac{C_9^3 \cdot 2! C_6^2 C_4^2}{2!} + \frac{C_9^2 C_7^2 C_5^2 C_3^2}{4!} = 22449$ permutations.

In conclusion, there are 27568 permutations satisfying the requirements.

Problem 139. *Let sequence $a_1, a_2, \cdots, a_{10}$ satisfy $1 \leqslant a_1 \leqslant la_2 \leqslant \cdots \leqslant a_{10} \leqslant 40$, $a_4 \geqslant 6$, and $\log_2(|a_i - i| + 1) \in \mathbf{N}(i = 1, 2, \cdots, 10)$ The number of such sequences is _____.*

Answer: 869

Reasoning: Consider $b_i = a_i - i$. Then $b_i \in \{0, \pm 1, \pm 3, \pm 7, \cdots\}$, and $b_{i+1} - b_i \geqslant -1$. Thus, the sequence $\{b_i\}$ has the following structure: there exists an integer $s \in [0, 10]$, $b_1, \cdots, b_s \in \{-1, 0, 1\}$, $3 \leqslant b_{s+1} \leqslant \cdots \leqslant b_{10}$ and any adjacent pair $b_1, \cdots, b_s$ is not 1 or -1.

Duo to $a_1 \geqslant 1, a_{10} \leqslant 4$ and $a_4 \geqslant 6$, it is known that $b_1 \neq -1$ and $b_{s+1}, \cdots, b_{10} \in \{3, 7, 15\}(0 \leqslant s \leqslant 3)$.

Conversely, for each sequence $\{b_i\}$, that meets the first two conditions, there exists a corresponding sequence $\{a_i\}$.

For a fixed $s \in \{0, 1, 2, 3\}$, the number of ways to choose , $b_1, \cdots, b_s$ is $1, 2, 5$ (because $(b_1, b_2) \neq (1, -1)$) and $5 \times 3 - 2 = 13$ (Because $(b_1, b_2, b_3) \neq (0, 1, -1), (1, 1, -1)$) .

Now let's find the number of ways to choose $b_{s+1}, \cdots, b_{10}$. Suppose there are x occurrence of 3, y occurrence of 7, z occurrence of 15. Then the non-negative integer solutions to $x+y+z = 10-s$ are C_{12-s}^2. Therefore the number of sequences $\{a_i\}$ is $1C_{12}^2 + 2C_{11}^2 + 5C_{10}^2 + 13C_9^2 = 869$.

Problem 140. *A student walks through a hallway with a row of closed lockers numbered from 1 to 1024. He starts by opening locker number 1, then proceeds forward, alternately leaving untouched or opening one closed locker. When he reaches the end of the hallway, he turns around and walks back, opening the first closed locker he encounters. The student continues this back and forth journey until every locker is opened. The number of the last locker he opens is _____.*

Answer: 342

Reasoning: Assuming there are 2^h closed lockers in a row, the number of the last locker opened by the student is denoted as a_k. When the student first reaches the end of the hallway, 2^{k-1} lockers remain closed. These closed lockers are all even-numbered, arranged in decreasing order from the student's standing position towards the other end. Now, these lockers are renumbered from 1 to 2^{k-1}. Note that the originally numbered locker n (where n is even) becomes locker number $2^{k-1} - \frac{n}{2} + 1$. According to the new numbering, the number of the locker the student opens last should be a_{k-1} , while the originally numbered a_k is now numbered as a_{k-1} . Therefore

$$\begin{aligned}&a_{k-1}=2^{k-1}-\tfrac{a_k}{2}+1\\&\Leftrightarrow a_k=2^k+2-2a_{k-1}\\&=2^k+2-2\left(2^{k-1}+2-2a_{k-2}\right)\\&=4a_{k-2}-2\\&\Rightarrow a_k-\tfrac{2}{3}=4\left(a_{k-2}-\tfrac{2}{3}\right)\end{aligned}$$

Given that $a_0=1$, and $a_1=2$, we have:

When k is even,
$a_k-\frac{2}{3}=\left(a_0-\frac{2}{3}\right)4^{\frac{k}{2}}\Rightarrow a_k=\frac{1}{3}\left(4^{\frac{k}{2}}+2\right)$;
When k is odd,, $a_k-\frac{2}{3}=\left(a_1-\frac{2}{3}\right)4^{\frac{k-1}{2}}\Rightarrow a_k=\frac{1}{3}\left(4^{\frac{k+1}{2}}+2\right)$.
Here [x] represents the greatest integer not exceeding x.

Thus $a_k=\frac{1}{3}\left(4^{\left[\frac{k+1}{2}\right]}+2\right)$ $(k\in\mathbf{N})$. For this problem, when $k=10, a_{10}=\frac{1}{3}\left(4^5+2\right)=342$, indicating that the last locker opened is numbered 342.

Problem 141. *Let $A=\{-3,-2,\cdots,4\}, a,b,c\in A$ be distinct elements of A. If the angle of inclination of the line: $ax+by+c=0$ is acute, then the number of such distinct lines is _____.*

Answer: 91
Reasoning: Because the angle of inclination of line ll is acute, ab¡0 . Counting is carried out in the following two cases:

(1) If a, b, and c are all not equal to 4, then $a,b,c\in\{-3,-2,\cdots,3\}$. Without loss of generality, assume $a>0$ and $b<0$. If $c=0$, there are 7 lines; if $c\neq 0$, there $3\times3\times4=36$ lines. In this case, there are a total of 43 distinct lines.

(2) If one of a, b, or c is 4. Let $a=4$, then $b<0$, there are 3 choices for b, and 6 choices for c, resulting in 18 lines (with duplicates). Among them, $4x-2y=0, 4x-2y+2=0$ are counted twice, so there are only 16 distinct lines. Similarly, when $b=4$ or $c=4$, there are also 16 distinct lines for each case. In conclusion, there are $43+163=9143+163=91$ distinct lines with acute angles of inclination.

Problem 142. *In a sequence of length 15 consisting of a and b, if exactly five "aa"s occur and both "ab" and "ba" and "bb" occur exactly three times, there are _____ such sequences.*

Answer: 980
Reasoning: Because there are three occurrences each of "ab" and "ba", there are two possible cases for such sequences:

(1).(a)(b)(a)(b)(a)(b)(a) ;

(2).(b)(a)(b)(a)(b)(a)(b) (here, (a) denotes a segment composed entirely of aa, and (b) denotes a similar segment composed entirely of b).

For case (1), since there are five "aa"s and three "bb"s, the sequence contains a total of 9 aa's and 6 bb's. The 9 a's can be partitioned into 4 segments, which equals the number of positive integer solutions to the equation $x_1+x_2+x_3+x_4=9$ which is $\mathrm{C}_{9-1}^{4-1}=\mathrm{C}_8^3$. Similarly, the 6 b's can be partitioned into 3 segments, which is $\mathrm{C}_8^3\mathrm{C}_5^2$.

For case (2), there are 8 aa's and 7 bb's, resulting in $\mathrm{C}_7^2\mathrm{C}_6^3$ possibilities.

Therefore, the total count is $\mathrm{C}_8^3\mathrm{C}_5^2+\mathrm{C}_7^2\mathrm{C}_6^3=980$.

Problem 143. *Among the five-digit numbers formed by the digits 1, 2, ..., 6, the number of five-digit numbers satisfying the condition of having at least three different digits and 1 and 6 not being adjacent is _____.*

Answer: 5880

Reasoning: Using a recursive approach, let S_n, denote the number of n-digit numbers formed by the digits $1,2,\cdots,6$ such that 1 and 6 are not adjacent.

Clearly, $S_1=6$ and $S_2=6^2-2=34$. Now we establish the recursive formula for S_n. We divide S_n into three categories:

Let a_n denote the number of nn-digit numbers with the first digit being 1, b_n denote the number of nn-digit numbers with the first digit being 2, 3, 4, or 5, and c_n

It's evident that $S_n=a_n+b_n+c_n$, and

$$\begin{aligned}
&a_n=a_{n-1}+b_{n-1},\\
&b_n=4\left(a_{n-1}+b_{n-1}+c_{n-1}\right),\\
&c_n=b_{n-1}+c_{n-1}.
\end{aligned}$$

Then $S_n=5\left(a_{n-1}+b_{n-1}+c_{n-1}\right)+b_{n-1}$

$$=5S_{n-1}+4S_{n-2}(n\geqslant 3).$$

From $S_1=6$ and $S_2=34$ and we get $S_3=194, S_4=1106, S_5=6306$.

Subtracting the cases where there is exactly one digit or exactly two digits, and 1 and 6 are not adjacent, from S_5, there are $\left(\mathrm{C}_6^2-1\right)\left(2^5-2\right)=420$.

Therefore, the number of five-digit numbers satisfying the condition is 6306-6-420=5880.

Problem 144. *How many numbers can be selected at most from 1 to 100 to ensure that the quotient of the least common multiple and greatest common divisor of any two numbers is not a perfect square?*

Answer: 61

Reasoning:

Let $a=mx$ and $b=my$ (x and y are coprime). Then we have $\frac{[a,b]}{(a,b)}=xy$. To ensure that xy is not a perfect square, x and y cannot both be square numbers. In other words, the two numbers cannot both be perfect squares after dividing them by their greatest common divisor. Based on this principle, construct the following sets: $(1,4,9,16,25,36,49,64,81,100),(2,8,18,32,50,72$ and there are 45 individual numbers. Selecting one number from each set, a maximum of $16+45=61$ numbers can be chosen to meet the requirement.

Problem 145. *From the natural numbers 1 to 100, choose any m numbers such that among these m numbers, there exists one number that can divide the product of the remaining $m-1$ numbers. The minimum value of m is _____.*

Answer: 26
Reasoning: First, when selecting 25 prime numbers, none of them can divide the product of the remaining 24 numbers. When $m = 25$, the condition cannot be satisfied, so m must be greater than 25.

Next, we prove that by selecting any 26 numbers from 1 to 100, there will always be one number that can divide the product of the remaining 25 numbers.

Let these 26 numbers be $a, b, c, \ldots, z$. Since there are only 25 prime numbers from 1 to 100, we can express these 26 numbers in their prime factorization form:

$$a = 2^{a_1} \times 3^{a_2} \times 5^{a_3} \times \cdots \times 97^{a_{25}}$$
$$b = 2^{b_1} \times 3^{b_2} \times 5^{b_3} \times \cdots \times 97^{b_{25}}$$
$$z = 2^{z_1} \times 3^{z_2} \times 5^{z_3} \times \cdots \times 97^{z_{25}}$$

From these 26 numbers, first, remove the one with the highest exponent of 2 (if there are ties, remove any one of them). Then, from the remaining 25 numbers, remove the one with the highest exponent of 3, then remove the one with the highest exponent of 5, and so on.

After removing 25 numbers, the remaining number will have exponents for each prime factor that do not exceed the maximum exponent among the 26 numbers. Therefore, this remaining number will definitely divide the product of the remaining 25 numbers.

Problem 146. *Several teams are participating in a friendly football match, where any two teams play at most one match against each other. It is known that each team has played 4 matches, and there are no draws. A team is considered a "weak team" if it loses at least 2 out of the 4 matches it plays. If there are only 3 "weak teams" in this friendly match, then at most how many teams could have participated in the matches?*

Answer: 9
Reasoning: Assume in a tournament with n teams, there are 3 weak teams, meaning the other n-3 teams are strong teams. There will be a total of $4n/2 = 2n$ games, resulting in 2n losses. Each weak team can lose up to 4 games, and each strong team can lose up to 1 game. Therefore, we have: $2n \leq 34 + n - 3$, from which we derive: $n \leq 9$. Hence, there can be at most 9 teams.

The construction is as follows: Teams 1, 2, 3, and 4 win against Team 7; Teams 3, 4, 5, and 6 win against Team 8; Teams 5, 6, 1, and 2 win against Team 9; Team 1 beats Team 2, Team 2 beats Team 3, Team 3 beats Team 4, Team 4 beats Team 5, Team 5 beats Team 6, and Team 6 beats Team 1; Teams 1 through 6, each with 3 wins and 1 loss, are the strong teams, while Teams 7, 8, and 9, each with 4 losses, are the weak teams.

Problem 147. *In the equation, the same letter represents the same digit, and different letters represent different digits. Then, the four-digit number $\overline{abcd}$ is given by:*

$$(\overline{ab})^c \times \overline{acd} = \overline{abcacd}$$

Answer: 3125
Reasoning: The original expression can be transformed using the principle of place value into:

$$(\overline{ab})^c \times \overline{acd} = \overline{abcacd}$$

According to this equation, where the same letter represents the same digit, and different letters represent different digits, let's analyze the possibilities:

First, we consider the case where $c = 3$:

Since $20^3 \times 200 = 1600000$, exceeding six digits, we conclude that $a = 1$. Thus, the equation becomes:

$$\left[(\overline{1b})^3 - 1\right] \times \overline{13d} = \overline{1b3} \times 1000$$

However, since there are 3 factors of 5 on the right side of the equation, and $\overline{13d}$ can only provide at most 1 factor of 5, we infer that $(\overline{1b})^3 - 1$ must be a multiple of 25. This implies that the last two digits of $(\overline{1})^3$ can be $01, 26, 51$, or 76. By considering the units place, we find that $b = 1$ or 6, but neither of these options satisfies the conditions.

Next, let's explore the case where $c = 2$. This yields:

$$\left[(\overline{ab})^2 - 1\right] \times \overline{a2d} = \overline{ab2000}$$

From this equation, we deduce that:

$$(\overline{ab})^2 - 1 = 1000 \times \frac{\overline{ab2}}{\overline{a2d}}$$

Because $0.5 < \frac{\overline{ab2}}{\overline{a2d}} < 2$, we conclude that $500 < (\overline{ab})^2 - 1 < 2000$. Hence, a can be $2, 3$, or 4. However, since $\overline{a2d}$ cannot be 125 or 625, it can provide at most 2 factors of 5. Therefore, $(\overline{ab})^2 - 1$ must be a multiple of 5, leading to the possible values $b = 1, 9, 4$, or 6.

Based on the equation $\left[(\overline{ab})^2 - 1\right] \times \overline{a2d} = \overline{ab2000}$, we have:

(1) $(\overline{ab} + 1) \times \overline{a2d} = \overline{ab2000}\,\mathrm{div}(\overline{ab} - 1) > 1000$

(2) $(\overline{ab} - 1) \times \overline{a2d} = \overline{ab2000}\,\mathrm{div}(\overline{ab} + 1) < 1000$

Consequently, a can only be 3. Then, $(\overline{ab} - 1) \times (\overline{ab} + 1)$ cannot be a multiple of 25, implying that $\overline{a2d}$ must be a multiple of 25. Thus, $d = 5$, and $\overline{a2d} = 325$.

Now, $(\overline{3b} - 1) \times (\overline{3b} + 1) \times 325 = \overline{3b2000}$, where $\overline{3b2000}$ is a multiple of 16. This means that either $(\overline{3b} - 1)$ or $(\overline{3b} + 1)$ must be a multiple of 8. Therefore, b can be 1 or 3. Since $a = 3$, b can only be 1.

After verifying: $31^2 \times 325 = 312325$, so $\overline{abcd} = 3125$.

Problem 148. *Let n be represented as the difference of squares of two nonzero natural numbers, then there are $F(n)$ ways to do so.*

For example, $15 = 8^2 - 7^2 = 4^2 - 1^2$, so $F(15) = 2$; whereas 2 cannot be represented, hence $F(2) = 0$. Then, the calculation result of $F(1) + F(2) + F(3) + \cdots + F(100)$ is ______.

Answer: 116

Reasoning: For a nonzero natural number a, if it can be expressed as the difference of squares of two nonzero natural numbers, let $a = m^2 - n^2$ (here m and n are both nonzero natural numbers, and $m > n$). Since $m^2 - n^2 = (m - n)(m + n)$, both $m - n$ and $m + n$ are factors of the natural number a. Considering that the parity of $m - n$ and $m + n$

is the same, thus for any pair of positive integers p and q with the same parity $(p > q)$, their product pq can always be expressed as the difference of squares of two nonzero natural numbers, $\left(\frac{p+q}{?}\right)^2 - \left(\frac{p-q}{?}\right)^2$ is the corresponding representation. This indicates that as long as we can find out how many pairs of positive integers (p, q) with the same parity exist such that $p > q$ and $1 \leq pq \leq 100$, the number of such pairs exactly equals the value of $F(1) + F(2) + F(3) + \cdots + F(100)$. Classification calculation:

When $q = 1$, p can take 3, 5, 7, 3, 99, there are 49 pairs (p, q) satisfying the condition;

When $q = 2$, p can take $4, 6, 8, \ldots, 50$, there are 24 pairs (p, q) satisfying the condition;

When $q = 3$, p can take 5, 7, 9, ...33, there are 15 pairs (p, q) satisfying the condition;

When $q = 4$, p can take $6, 8, 10, \ldots, 24$, there are 10 pairs (p, q) satisfying the condition;

When $q = 5$, p can take $7, 9, 11, \ldots, 19$, there are 7 pairs (p, q) satisfying the condition;

When $q = 6$, p can take $8, 10, 12, 14, 16$, there are 5 pairs (p, q) satisfying the condition;

When $q = 7$, p can take $9, 11, 13$, there are 3 pairs (p, q) satisfying the condition;

When $q = 8$, p can take $10, 12$, there are 2 pairs (p, q) satisfying the condition;

When $q = 9$, p can only take 11, there is only 1 pair (p, q) satisfying the condition; When $q > 9$, there are no pairs (p, q) satisfying the condition.

In conclusion, there are a total of 116 pairs satisfying the condition, which is the desired result.

Problem 149. *What is the simplified value of the expression,* $8x^3 - 3xy + \sqrt{p}$*, if* $p = 121$*,* $x = -2$*, and* $y = \frac{3}{2}$*?*

A) 84 *B)* $73 + \sqrt{11}$

C) -28 *D)* -44

Answer: D

Reasoning: By plugging in the variables for x, y, and z, we arrive at the equation $8 \times (-2)3 - 3 \times (-2) \times 32 + 121$ which evaluates to $-64 + 9 + 11$, leading to the answer -44.

Problem 150. *Which expression best represents "the product of twice a quantity x and the difference of that quantity and 7"?*

A) $2x(7 - x)$ *B)* $2x(x - 7)$

C) $2x - (x - 7)$ *D)* $2(x - 7)$

Answer: B

Reasoning: The phrase "the product of" refers to multiplying the next two groupings mentioned. In this case, our next two groupings are "twice a quantity x" and "the difference of that quantity and 7". We can evaluate this first group as 2x and the second group as $(x - 7)$. Multiplying the two results in $2x(x - 7)$.

Problem 151. *The formula for the area of a triangle is* $A = \frac{1}{2}bh$*. The area of a triangle is 62 square meters, and its height is 4 meters. What is the length of the base?*

Options:
A) 15.5 m
B) 27 m
C) 31 m
D) 62 m

Answer: C
Reasoning: Given that the area is 62 square meters and the height is 4 meters, we can arrive at the equation $62 = \frac{1}{2}b \cdot 4$. This can be simplified to $62 = 2b$ and further to $b = 31$.

Problem 152. *Simplify:* $(-m^2n^{-3})^3 \cdot (4m^{-1}n^2p^3)^2$

A) $\dfrac{-16m^4p^6}{n^5}$	*B)* $\dfrac{16m^3p^5}{n^2}$
C) $\dfrac{8m^3p^5}{n^2}$	*D)* $\dfrac{-16m^6p^9}{n^{23}}$

Answer: *A*
Reasoning: First simplify the expression by expanding the exponents through the equation: $(-m^6n^{-9}) \cdot (16m^{-2}n^4p^6)$. Combining like terms results in $-16 \times m^4 \times n^{-5} \times p^6$.

Problem 153. *If* $M = 24a^{-2}b^{-3}c^5$ *and* $N = 18a^{-7}b^6c^{-4}$*, then* $\frac{N}{M} =$

A) $\dfrac{4a^5c^9}{3b^9}$	*B)* $\dfrac{4a^5c}{3b^3}$
C) $\dfrac{3b^9}{4a^5c^9}$	*D)* $\dfrac{3b^3}{4a^5c}$

Answer: *C*
Reasoning: Using law of exponents and combining like terms arrives at answer C.

Problem 154. *The volume of a rectangular prism is* $2x^5 + 16x^4 + 24x^3$*. If the height of the rectangular prism is* $2x^3$*, which of the following could represent one of the other dimensions of the rectangular prism?*

A) $(x + 2)$	*B)* $(x + 3)$
C) $(x + 4)$	*D)* $(x + 12)$

Answer: *A*
Reasoning: we can first divide the volume of the rectangular prism by $2x^3$ to get $x^2+8x+12$. Factoring this expression results in $(x + 6)(x + 2)$. Therefore, the answer is A

Problem 155. *Which expression is equivalent to* $(3x - 1)(2x^2 + 1)$*?*
Options:
A) $6x^3 - 2x^2 - 3x + 1$

B) $6x^3 - 2x^2 + 3x - 1$
C) $5x^3 - 2x^2 + 3x - 1$
D) $5x^3 - x^2 + 4x$

Answer: B
Reasoning: Multiplying out the expression with FOIL (first outer inner last) nets $6x^3 + 3x - 2x^2 - 1$.

Problem 156. *Factor* $64b^2 - 16b + 1$ *completely.*
Options:
A) $(32b - 1)(32b + 1)$
B) $(b - 8)^2$
C) $(8b - 1)^2$
D) $(8b + 1)^2$

Answer: C
Reasoning: We first observe that this follows the pattern for a perfect square binomial. This leads us to the answer $(8b - 1)^2$.

Problem 157. *Simplify* $6\sqrt[3]{64} - \sqrt{12} \cdot 2\sqrt{27}$

A) 12 | *B)* -12
C) $24 - 12\sqrt{3}$ | *D)* $48 - 72\sqrt{3}$

Answer: *B*
Reasoning: $6\sqrt[3]{64}$ can be simplified into 24 by basic arithmetic rules. $12 \cdot 2\sqrt{27}$ can be reduced to $2\sqrt{324}$ which is equivalent to 36. Finally $24 - 36 = -12$.

Problem 158. *What is the simplified form of* $\sqrt{80x^5y^2z^3}$*? Assume all variables are positive.*

A) $16x^2yz\sqrt{5xz}$ | *B)* $16xyz\sqrt{5x^3z}$
C) $4x^2yz\sqrt{5xz}$ | *D)* $4\sqrt{5x^5y^2z^3}$

Answer: *C*
Reasoning: Take out any variable or constant that has a squared factor: $4x^2yz\sqrt{5xz}$

Problem 159. *Which of the following is NOT equivalent to* -4*?*

A) $2\sqrt{9} - 5\sqrt[3]{8}$ | *B)* $3\sqrt[3]{64} - 2\sqrt{64}$
C) $2\sqrt{121} - 3\sqrt[3]{216}$ | *D)* $4\sqrt{25} - 8\sqrt[3]{27}$

Answer: *C*
Reasoning: A,B,D all equivalent to -4 except C leads to 3.

Problem 160. *What are the solutions of the quadratic equation* $15x^2 = 2x + 8$

A) $\{-\frac{4}{3}, -\frac{3}{2}\}$ *B)* $\{-\frac{4}{5}, \frac{2}{3}\}$

C) $\{-\frac{3}{2}, \frac{4}{5}\}$ *D)* $\{-\frac{2}{3}, \frac{4}{5}\}$

Answer: *D*
Reasoning: First move all terms to one side: $15x^2 - 2x - 8 = 0$. Then factor into $(5x - 4)(3x + 2) = 0$. Setting $5x - 4$ to zero results in a solution of $x = \frac{4}{5}$ and setting $3x + 2$ to zero results in a solution of $x = -\frac{2}{3}$.

Problem 161. *Find the solution of* $4(3y - 5) = 2(7y + 3)$

A) -13 *B)* -4

C) $\frac{11}{2}$ *D)* 13

Answer: *A*
Reasoning: First distribute and expand to $12y - 20 = 14y + 6$. Combine like terms to arrive at $2y = -26$. Thus, $y = -13$.

Problem 162. *Solve the equation* $x^2 - 7 + 2x = 0$

A) $\{-1 - 2\sqrt{2}, -1 + 2\sqrt{2}\}$ *B)* $\{1 - 2\sqrt{2}, 1 + 2\sqrt{2}\}$

C) $\{\frac{-7 - \sqrt{41}}{2}, \frac{-7 + \sqrt{41}}{2}\}$ *D)* $\{\frac{7 - \sqrt{41}}{2}, \frac{7 + \sqrt{41}}{2}\}$

Answer: *A*
Reasoning: Utilize the quadratic formula with $a = 1$, $b = 2$, and $c = -7$. $\frac{-2 \pm \sqrt{2^2+28}}{2} = -1 \pm \sqrt{8} = -1 \pm 2\sqrt{2}$.

Problem 163. *At a movie theater, the adult ticket price is \$8 and the child ticket price is \$6. For a certain movie, 210 tickets were sold and \$1500 was collected. How many adult tickets were sold?*

A) 30 *B)* 90

C) 120 *D)* 150

Answer: *C*
Reasoning: We can create a system of equations $8x + 6y = 1500$ and $x + y = 210$. By rearranging the second equation to $y = 210 - x$ and plugging $210 - x$ in for y in the first equation, we get 8x+1260-6x = 1500 which evaluates to $2x = 240$. Therefore $x = 120$ and $y = 90$.

Problem 164. *Solve for* $y : x - yz + 5 = 8$

A) $y = \dfrac{x-3}{z}$ B) $y = \dfrac{x-13}{z}$

C) $y = \dfrac{3-x}{z}$ D) $y = \dfrac{13-x}{z}$

Answer: *A*
Reasoning: Attempt to isolate y by first isolating the yz term: $yz = x - 3$. Then divide by z for $y = \frac{x-3}{z}$.

Problem 165. *Describe the type of solution for the linear system of equations defined by*

$$\begin{cases} 2y - 3x &= 20 \\ -\frac{3}{2}x + y &= 10 \end{cases}$$

Options:
A) no solution
B) infinite solutions
C) one solution
D) two solutions

Answer: B
Reasoning: First, rearrange the second equation to $y = \frac{3}{2}x + 10$. Then plug this into the first equation for $3x+20-3x = 20$. This is obviously true so there must be infinite solutions.

Problem 166. *Identify all of the following equations that have a solution of -2?*

I) $3(x + 7) = 5(x + 5)$ II) $x^2 + x - 6 = 0$

III) $2(x - 4) = x - 10$ IV) $x^2 = 4$

A) $IandIII$ B) $IIandIV$

C) $I, IIandIV$ D) $I, IIIandIV$

Answer: *D*
Reasoning: Plug in -2 for x in each equation and see if it works I: $3 \times 5 = 5 \times 3$ works.
II: $4 - 2 - 6 = 0$ does not work.
III:$2 \times -6 = -2 - 10$ works.
IV: $4 = 4$ works.

Problem 167. *What is the solution of the system of linear equations?*

$$\begin{cases} 7x - 2y = 5 \\ -3x - 4y = 7 \end{cases}$$

A) $(\frac{3}{17}, -\frac{32}{17})$ *B)* $(-\frac{3}{17}, -\frac{32}{17})$

C) $(\frac{3}{17}, \frac{32}{17})$ *D)* $(-\frac{3}{17}, \frac{32}{17})$

Answer: *A*

Reasoning: We will use the elimination method to get rid of y by multiplying the first equation by 2 and subtracting the two equations from each other. This results in $17x = 3$ so $x = \frac{3}{17}$ and $y = -\frac{32}{17}$.

Problem 168. *What values of x make the inequality true?* $4(x-2) - 10x \geq -3x + 13$

A) $\{x : x \geq 1\}$ *B)* $\{x : x \geq -7\}$

C) $\{x : x \leq 1\}$ *D)* $\{x : x \leq -7\}$

Answer: D

Reasoning: Simplify the inequality into $-6x-8 \geq -3x+13$. This is equivalent to $-3x \geq 21$ so $x \leq -7$.

Problem 169. *What is the equation of the line that passes through the points* $(4, -4)$ *and* $(-5, 14)$*?*

A) $x + 2y = 2$ *B)* $2x + 3y = 12$

C) $2x + y = 4$ *D)* $3x - 2y = -6$

Answer: C

Reasoning: First find slope $\frac{14+4}{-5-4} = -2$, then use point-slope form on one of the points. $y + 4 = -2(x - 4)$. This simplifies to $y = -2x + 4$. Convert to standard form.

Problem 170. *The cost c per person to participate in a guided mountain biking tour depends on the number of people n participating in the tour. This relationship can be described by the function $c = -3n + 60$, where $0 < n < 12$. What is the rate of change described by this function?*

A) 20 people per tour *B) -3 people per tour*

C) 20$ per person *D) -3$ per person*

Answer: D

Reasoning: $c = -3n+60$ can be interpreted as the cost per person $= -3\times$number of people$+$ 60. So as more people join the cost per person decreases by 3 dollars per person joining.

Problem 171. *Find $f(-3)$ if $f(x) = 6x^2 - x - 2$.*

A) 55 *B)* 53

C) -53 *D)* -59

Answer: A
Reasoning: Plug in -3 for x to arrive at $6 \times (-3)^2 + 3 - 2 = 55$

Problem 172. *When Darcy's school bus travels at 30 miles per hour, it gets from her home to school in 12 minutes. What is the speed of Darcy's bus if it makes the same trip in 18 minutes?*

A) 20 *mph*	*B)* 28 *mph*
C) 36 *mph*	*D)* 45 *mph*

Answer: A
Reasoning: If the bus travels for 12 minutes at 30 mph, the bus travels $30 \times 12/60$ miles or 6 miles. If the bus does the same trip in 18 minutes, it would be at a speed of $6/(18/60)$ or 20 miles per hour.

Problem 173. *The price of a package varies directly with the number of stickers in the package. If a package contains 650 stickers and sells for $26.00, what is the constant of variation? How much will 800 stickers cost?*

A) $k = 0.04; \$32.00$	*B)* $k = 0.40; \$320.00$
C) $k = 6.24; \$806.24$	*D)* $k = 25; \$20,000.00$

Answer: A
Reasoning: Find the constant rate by taking $26/650 = 0.04$. Then 800×0.04 results in $32.

Problem 174. *Which function does NOT have an x-intercept?*

A) $y = \frac{1}{2}x - 7$	*B)* $y = -\frac{1}{3}x - 5$
C) $y = -x^2 + 2x + 5$	*D)* $y = x^2 - 2x + 5$

Answer: D
Reasoning: A and B are both linear equations with a non zero slope so they must have an x intercept. Plug in $y = 0$for the other two equations and attempt to solve a solution. D results in no solutions.

Problem 175. *What is the x-intercept and Y-intercept of the graph of* $5x–3y = -30$*?*

A) The x-intercept is 6, and the y-intercept is -10
B) The x-intercept is -6, and the y-intercept is 10.
C) The x-intercept is 10, and the y-intercept is -6.
D) The x-intercept is -10, and the y-intercept is 6.

Answer: B
Reasoning: Plug in $y = 0$ to solve for x intercept: $5x - 3 \times 0 = -30,\ x = -6$. Plug in $x = 0$ to solve for y intercept: $5 \times 0 - 3y = -30,\ y = 10$.

Problem 176. *Is the point* $(1, -3)$ *a solution to the equation* $f(x) = x^2 + 4x–8$*?*

A) Yes *B) No*

Answer: A
Reasoning: Plug in the point: $-3 = 12 + 4 \times 1 - 8$. True.

Problem 177. *If y varies inversely with x, and* $x = 18$ *when* $y = 4$*, find y when* $x = 12$*.*

A) $y = 3$	*B)* $y = 6$
C) $y = 9$	*D)* $y = 54$

Answer: B
Reasoning: Inverse variation means that $x \times y = c$. here $18 \times 4 = 72$ so we know our $c = 72$. Now we can plug in $x = 12$ for $12 \times y = 72$. We get $y = 6$.

Problem 178. *If 10 workers can build a house in 12 weeks, how long will it take 15 workers to build the same house?*

A) 6	*B)* 16
C) 8	*D)* 18

Answer: C
Reasoning: First calculate total weeks to build a house. With 10 workers and 12 weeks, we have a total of 120 weeks. Taking 120 and dividing by 15 workers results in 8 weeks needed.

Problem 179. *Write the equation of the line that has a y–intercept of –3 and is parallel to the line* $y = -5x + 1$*.*

Answer: $y = -5x - 3$
Reasoning: Parallel lines have the same slope but different y-intercepts.

Problem 180. *A model house was built that states that 3 inches represents 10 ft. If the width of the door on the model is 1.2 inches, what is the width of the actual door?*

Answer: 4ft
Reasoning: Use proportional representation to set up $3in/10ft = 1.2in/Xft$, cross multiplication $3x = 12, x = 4ft$.

Problem 181. *The product of 4 more than a number and 6 is 30 more than 8 times the number. What is the number?*

Answer: -3
Reasoning: $6(x+4) = 8x+30 \Rightarrow 6x+24 = 8x+30 \Rightarrow 2x = -6 \Rightarrow x = -3$

Problem 182. *Solve* $4(x+4) = 24 + 3(2x-2)$.

Answer: $x = -1$
Reasoning: Use distributive property to both sides of the equations, and then simplify to $2x = -2$, then $x = -1$.

Problem 183. *Three times as many robins as cardinals visited a bird feeder. If a total of 20 robins and cardinals visited the feeder, write a system of equations to represent the situation and solve how many were robins?*

Answer: 5 Robins.
Reasoning: Define x as the number of robins, and define y as the number of cardinals, then set up system of equations. $x + y = 20$, $y = 3x$ Use substitution method to solve, $x + 3x = 20$, $x = 5$.

Functions and Applications

Problem 184. *Describe all the transformations of the function:* $f(x) = -|x-3| + 1$

A) Translated 3 units left, 1 unit up and reflected over the x-axis
B) Translated 1 unit right, 3 units down and reflected over the x-axis
C) Translated 3 units right and 1 unit up and reflected over the y-axis
D) Translated 3 units right and 1 unit up and reflected over the x-axis

Answer: D
Reasoning: With the parent function as the absolute value function, we can see that the +1 causes a one unit translation up and the x-3 causes a 3 unit translation right. Because there is a negative sign in front of the absolute value there is also a reflection over the x axis

Problem 185. *If the value of the discriminant for the function* $f(x) = 2x^2 - 5x + 6$ *is equal to -23, which of the following correctly describes the graph of* $f(x)$*?*

I) $f(x)$ *has real roots.* *II)* $f(x)$ *has imaginary roots.*
III) $f(x)$ *has two solutions.* *IV)* $f(x)$ *has one solution.*

A) I and III only *B) I and IV only*
C) II and III only *D) IV only*

Answer: C
Reasoning: Since our discriminant is negative there must be imaginary roots and two solutions.

Problem 186. *What is the axis of symmetry of the function* $y = -3(x+1)^2 + 4$*?*

A) $x = -3$	*B)* $x = -1$
C) $x = 1$	*D)* $x = 4$

Answer: *B*
Reasoning: This function is a parabola and there is a shift of 1 to the left, this makes the axis of symmetry at $x = -1$.

Problem 187. *Which equation shows the function* $f(x) = 12x^2+36x+27$ *in intercept form?*

A) $f(x) = 3(2x-3)^2$	*B)* $f(x) = 3(2x+3)^2$
C) $f(x) = (6x-9)(2x-3)$	*D)* $f(x) = 3(2x+3)(2x-3)$

Answer: *B*
Reasoning: Factoring 3 out of the function results in $f(x) = 3(4x^2 + 12x + 9)$. This is a perfect square of $2x + 3$.

Problem 188. *Which of the following represents the function in intercept form* $y = 64x^2 - 49$*?*

A) $y = (64x+1)(x-49)$	*B)* $y = (8x+7)(8x-7)$
C) $y = (8x+7)(8x+7)$	*D)* $y = (8x-7)(8x-7)$

Answer: *B*
Reasoning: Since the function is in the form $S^2 - s^2$, we can simplify to $(S+s)(S-s)$.

Problem 189. *Use factoring to find the solutions to the equation* $x^2 + 24x = -144$.

A) -12	*B)* 12
C) -12 *and* 12	*D)* 9 *and* 16

Answer: *A*
Reasoning: $x^2 + 24x + 144 = 0$, then $(x+12)^2 = 0$.

Problem 190. *Solve the equation* $-x^2 - 11 = -2x^2 + 5$ *for the variable* x.

A) ± 2	*B)* ± 4
C) $\pm\sqrt{2}$	*D)* $\pm\sqrt{6}$

Answer: *B*
Reasoning: Rearrange equation to $x^2 - 16 = 0$. solving results in $x = \pm 4$.

Problem 191. *What must be added to the equation $x^2 + 20x = 0$ to complete the square?*
Options:
A) 10
B) 25
C) 40
D) 100

Answer: D
Reasoning: For $x^2 + 20x + c$ to be a complete square, c must be $\left(\frac{20}{2}\right)^2 = 100$.

Problem 192. *If $f(x) = x^2 + 4$ and $g(x) = \sqrt{10 - x}$, what is the value of $f(g(1))$?*

A) 1 *B)* 0
C) $\sqrt{5}$ *D)* 13

Answer: *D*
Reasoning: first plug 1 in as x for the g equation, $g(1) = 3$. Then plug 3 in for x in the f function, $f(3) = 13$.

Problem 193. *Find all the solutions to the function:* $0 = (-4x + 9)(x - 1)(3x - 5)$

A) $x = -\frac{7}{4}, x = 1, x = \frac{7}{3}$ *B)* $x = \frac{9}{4}, x = 1, x = \frac{5}{3}$
C) $x = 9, x = -1, x = -\frac{5}{3}$ *D)* $x = \frac{9}{8}, x = \frac{1}{2}, x = \frac{5}{6}$

Answer: *B*
Reasoning: By setting each part in parenthesis to zero and solving, we can find the three solutions.

Problem 194. *What are the solutions to the equation $0.5x^2 - 0.45x - 0.3 = 0$?*

A) $x = -1.35$ *and* $x = 0.45$ *B)* $x = 1.35$ *and* $x = -0.45$
C) $x = -1.35$ *and* $x = -0.45$ *D)* $x = 1.35$ *and* $x = 0.45$

Answer: *B*
Reasoning: Plug into the quadratic formula.

Problem 195. *Find the remainder when $f(x) = 5x^4 + 2x^2 - 3x + 1$ is divided by $x - 2$.*

A) 95 *B)* 43
C) 83 *D)* -25

Answer: *C*
Reasoning:

$$\begin{aligned}\frac{5x^4+2x^2-3x+1}{x-2} &= 5x^3+\frac{10x^3+2x^2-3x+1}{x-2}\\ &= 5x^3+10x^2+\frac{22x^2-3x+1}{x-2}\\ &= 5x^3+10x^2+22x+\frac{41x+1}{x-2}\\ &= 5x^3+10x^2+22x+41+\frac{83}{x-2}\end{aligned}$$

Problem 196. *The path of an object falling to Earth is represented by the equation* $h(t) = -16t^2 + vt + s$. *What is the equation of an object that is shot up into the air from 150 feet above the ground and has an initial velocity of 62 feet per second?*

A) $h(t) = -16t^2 + 62t + 150$ *B)* $h(t) = -16t^2 + 150t + 62$
C) $h(t) = -16t^2 + 62t$ *D)* $h(t) = -16t^2 + 150t$

Answer: *A*
Reasoning: Plug in 150 for s and 62 for v.

Problem 197. *The graph above shows a portion of a system of equations where* $f(x)$ *has* $a > 0$ *and* $g(x)$ *has* $a < 0$. *Which of the following satisfies the equation* $f(x) = g(x)$*?*

A) $\{(0,-1);(-3,2)\}$ *B)* $\{(-1,2);(-2,3)\}$
C) $\{(1,2);(2,7)\}$ *D)* $\{(-3.7,0);(0.4,0)\}$

Answer: *A*
Reasoning: Only the points of intersection satisfy the equation.

Problem 198. *Which of the following is the conjugate of the expression* $\frac{2}{3-\sqrt{2}}$*?*

A) $3-\sqrt{2}$ *B)* $3+\sqrt{2}$
C) $\sqrt{2}$ *D)* $-\sqrt{2}$

Answer: *B*
Reasoning: The conjugate is changing the sign in the denominator.

Problem 199. *Solve the equation* $\frac{2}{x+5} + \frac{3}{x-5} = \frac{7x-9}{x^2-25}$ *for the variable* x.

A) $x = 5$ *B)* $x = \frac{9}{2}$
C) $x = 2$ *D)* $x = 7$

Answer: D
Reasoning: $\frac{2(x-5)+3(x+5)}{x^2-25} = \frac{7x-9}{x^2-25}.5x+5 = 7x-9.x = 7$

Problem 200. *Choose all the following that have an end behavior as $x \to \infty, f(x) \to \infty$. NOTE: You may choose more than one.*

A) $f(x) = -x^4 + 3x^2 - x - 7$
B) $f(x) = x^3 + 5x + 1$
C) $f(x) = -3x^5 + 2x^3 + 9x - 4$
D) $f(x) = -2(x-7)^2 + 4$
E) $f(x) = -3x + 2$
F) $f(x) = (x-8)^2 + 2$

Answer: B, F
Reasoning: Functions with the desired end behavior must either be positive even functions or positive odd functions.

Problem 201. *Which of the following is true about the function $f(x) = x^5 + 3x^4 + 9x^3 - 23x^2 - 36$?*

I) $f(x)$ *has five real roots.*

II) $f(x)$ *has three imaginary roots.*

III) $f(x)$ *has a double root.*

IV) As $x \to \infty, f(x) \to \infty$

Options:
A) I, II, IV
B) II and III
C) III and IV
D) II and IV

Answer: C
Reasoning: We know 4 is true based on the fact that this is a positive odd polynomial. Using Descartes' rule of signs we see that we can have maximum 1 positive real root and 4,2, or 0 negative real roots. By doing some factoring we find that we have a double root.

Problem 202. *Simplify the expression:* $\frac{3x^2}{2x^{-1}} \cdot \frac{y^2}{6x^2y}$

A) $\dfrac{xy}{4}$
B) $\dfrac{4x}{y}$
C) $\dfrac{y}{4x}$
D) $\dfrac{4}{xy}$

Answer: A
Reasoning: Cancel out like terms with exponent rules.

Problem 203. *Solve the equation* $-2(3x+2)^{\frac{3}{2}} = -54$ *for the variable* x.

A) $x = 9$ B) $x = \frac{5}{6}$

C) $x = \frac{23}{6}$ D) $x = \frac{7}{3}$

Answer: D
Reasoning: $(3x+2)^{3/2} = 27, 3x+2 = 9, 3x = 7, x = 7/3$

Problem 204. *Where is the hole in the graph of the function* $f(x) = \frac{x^2+x-6}{x+3}$*?*

A) $(3, 1)$ B) $(-3, 0)$

C) $(-3, -6)$ D) $(-3, -5)$

Answer: D
Reasoning: Hole at x = -3 because that value creates a division by zero. By dividing both sides by $x+3$ we are left with $x-2$, so plugging -3 results in a y value of -5.

Problem 205. *What is the horizontal asymptote of the function* $f(x) = \frac{2x^3+2}{x^2-16}$*?*

A) $y = 4$ *and* $y = 4$ B) $y = 2$

C) $y = 0$ D) *no horizontal asymptotes*

Answer: D
Reasoning: Because the exponent in the numerator is larger than the one in the denominator there is no asymptote.

Problem 206. *Determine any domain restriction(s) given the expression* $\frac{x^2-9}{x^2-3x-18}$.

A) $x = 3$ *and* $x = -3$ B) $x = 6$ *and* $x = -3$

C) $x = 6$ D) $x = \frac{1}{2}$

Answer: B
Reasoning: $\frac{(x+3)(x-3)}{(x-6)(x+3)}$, x cannot be -3 due to a hole and 6 due to an asymptote.

Problem 207. *Simplify the rational expression* $\frac{n^2+2n-24}{n^2-11n+28}$

A) $\frac{n+6}{n-7}$ B) $\frac{n+6}{n-4}$

C) $\frac{n+6}{n+7}$ D) $\frac{n-4}{n-7}$

Answer: A
Reasoning: $\frac{(n+6)(n-4)}{(n-7)(n-4)} = \frac{n+6}{n-7}$

Problem 208. *Simplify the rational expression.*

$$\frac{\frac{x+8}{x^2-64}}{(x+8)(x-8)}$$

Options:
A) $x+2$
B) $\frac{4(x+8)}{(x^2-64)^2}$
C) $\frac{x+8}{4}$
D) $\frac{1}{4}$

Answer: C
Reasoning: the numerator can first be simplified into $\frac{1}{x-8}$. This allows the whole fraction to be simplified into $\frac{x+8}{4}$.

Problem 209. *Simplify the expression:* $5\sqrt{2} - 3\sqrt{8} + 2\sqrt{18}$.
Options:
A) $5\sqrt{2}$
B) $-\sqrt{2} + 6\sqrt{3}$
C) $5\sqrt{2} - 12\sqrt{2} + 6\sqrt{2}$
D) $-\sqrt{2}$

Answer: A
Reasoning: $5\sqrt{2} - 6\sqrt{2} + 6\sqrt{2} = 5\sqrt{2}$

Problem 210. *Which expression is the simplest form of* $4\sqrt[3]{32} - \sqrt[3]{32}$ *?*

A) $3\sqrt[3]{4}$ *B)* $6\sqrt[3]{4}$
C) $3\sqrt[3]{32}$ *D)* $16\sqrt[3]{2} - 4$

Answer: B
Reasoning: $\sqrt[3]{32} = 2\sqrt[3]{4}, 3 \times 2\sqrt[3]{4} = 6\sqrt[3]{4}$

Problem 211. *What is the simplified form of the expression* $\sqrt{98x^3y^5z}$ *?*

A) $2xyz\sqrt{7xyz}$ *B)* $7x^2y^2\sqrt{2yz}$
C) $7xy^2\sqrt{2xyz}$ *D)* $49xy^2\sqrt{2xyz}$

Answer: C
Reasoning: Take out all square factors.

Problem 212. *Evaluate each of the following expressions. a)* $\log_4 \frac{1}{64} =?$ *b)* $\log_5 625 =?$

Answer: a. 3, b. 4
Reasoning: a. $4^x = 1/64, x = 3$, b. $5^x = 625, x = 4$

Problem 213. *Jasmine invests* \$2,658 *in a retirement account with a fixed annual interest rate of* 9% *compounded continuously. What will the account balance be after 15 years?*

Answer:\$ 9681.72
Reasoning: Using the compound interest formula we have $2658 \times (1.09)^{15} = 9681.72$

Problem 214. *Remy invests* \$8,589 *in a retirement account with a fixed annual interest rate of* 7% *compounded continuously. How long will it take for the account balance to reach* \$21,337.85 *?*

Answer: 13.45 years
Reasoning: Using the compound interest formula we have $8589 \times (1.07)^x = 21337.85, x = 13.45$

Problem 215. *Solve* $\sqrt{x^2 + 2x - 6} = \sqrt{x^2 - 14}$

Answer: -4
Reasoning: Square both sides then rearrange to $2x = -8, x = -4$.

Problem 216. *In* $\triangle \mathrm{ABC}, \mathrm{AB} = 10$ cm, $\angle \mathrm{B} = 90°$, *and* $\angle \mathrm{C} = 60°$. *Determine the length of* BC.

A) 10 cm
B) $10\sqrt{3}$ cm
C) $\dfrac{10\sqrt{3}}{3}$ cm
D) 20 cm

Answer: C
Reasoning: This is a special $30 - 60 - 90$ triangle where AB is the longer leg and BC is the shorter leg. To go from the longer leg to the shorter one we should multiply by $\frac{\sqrt{3}}{3}$

Problem 217. *If the length of the shorter leg of a* $30° - 60° - 90°$ *triangle is* $5\sqrt{3}$, *then the length of the longer leg is*

A) 10
B) $10\sqrt{3}$
C) $10\sqrt{6}$
D) 15

Answer: D
Reasoning: To find the longer leg of a 30-60-90 triangle from the shorter leg, we must multiply by $\sqrt{3}$. This means our longer leg has a length of 15

Problem 218. *If the sides of a triangle are 6,7 , and 9 ; then the triangle is*

A) a $45° - 45° - 90°$ *triangle*
B) an acute triangle
C) an obtuse triangle
D) a right triangle

Answer: B
Reasoning: We can eliminate A because there are not two equal sides. Now compare 6^2+7^2 and 9^2. Since 6^2+7^2 is larger, this is an acute triangle.

Problem 219. *The _____ ratio compares the length of the adjacent leg to the length of the hypotenuse*

A) sine	*B) cosine*
C) tangent	*D) none of the above*

Answer: *B*
Reasoning: Cosine compares adjacent and hypotenuse.

Problem 220. *Which of the following forms a right triangle?*

A) $\sqrt{4}, \sqrt{9}, \sqrt{25}$	*B)* $1, 2, 3$
C) $5, 11, 13$	*D)* $3, 4, 5$

Answer: *D*
Reasoning: D is a 3-4-5 triangle which is a known right triangle. This problem can also be done by using the pythagorean theorem.

Problem 221. *Find the length of the diagonal of a square whose perimeter measures 28 cm.*

A) 7 cm	*B)* $7\sqrt{2}$ *cm*
C) $7\sqrt{3}$ *cm*	*D)* $28\sqrt{2}$ *cm*

Answer: *B*
Reasoning: A square has 4 equal sides so each side must be 7 cm. The length of the diagonal is found by a 45-45-90 triangle so the diagonal is $7\sqrt{2}$ cm.

Problem 222. *Which of the following transformations creates a figure that is similar (but not congruent) to the original figure?*

A) Dilation	*B) Rotation*
C) Translation	*D) Reflection*

Answer: *A*
Reasoning: B and C don't change the shape of the object and D results in a nonsimilar object. Dilation keeps similarity while changing size.

Problem 223. *What is the image of the point (4, –2) after a dilation of 3?*

A) $(12, -6)$ *B)* $(7, 1)$

C) $(1, -5)$ *D)* $(\frac{4}{3}, -\frac{2}{3})$

Answer: *A*
Reasoning: A dilation of 3 means we should multiply each coordinate by 3.

Problem 224. *What is the center of the circle whose equation is* $(x-1)^2 + (y+3)^2 = 25$*?*

A) $(-1, 3)$ *B)* $(3, -1)$
C) $(1, -3)$ *D)* $(-3, 1)$

Answer: *C*
Reasoning: The center is (h, k) with circle equation of $(x-h)^2 + (y-k)^2 = r^2$

Problem 225. *A _____ is a quadrilateral with two pairs of congruent adjacent sides and no congruent opposite sides.*

A) Rectangle *B) Rhombus*
C) Kite *D) Trapezoid*

Answer: *A*
Reasoning: Rectangles and Rhombus both have congruent opposite sides and a Trapezoid does not have two pairs of congruent adjacent sides. The Kite is the only thing that fits the description.

Problem 226. *Find the number of sides of a convex polygon if the measures of its interior angles have a sum of* $2340°$*.*

A) 13 *B)* 11
C) 15 *D)* 7

Answer: *C*
Reasoning: A convex polygon has interior angles summing to $180(n-2)$ where n is the number of sides. Solving for n results in 15 sides.

Problem 227. *The base of a square pyramid has sides of 10 and the slant height is 15. Find the surface area of the pyramid.*

A) 85 *B)* 220
C) 310 *D)* 400

Answer: *D*
Reasoning: If the slant of the pyramid is 15 and the sides are 10, each side triangle has an area of 75 and the base has an area of 100. This comes to a sum of 400.

Problem 228. *Find the volume of a cone, to the nearest cubic inch, whose radius is 12 inches and whose height is 15 inches.*

A) 2827	*B)* 2262
C) 565	*D)* 188

Answer: *B*
Reasoning: The area of a cone is $\frac{1}{3}bh$, since radius is 12 the base is 144π.

Problem 229. *Find the volume of a hemisphere (half a sphere) whose radius is 10 feet. Round the answer to the nearest cubic foot.*

A) 419	*B)* 1047
C) 2094	*D)* 4189

Answer: *C*
Reasoning: The formula for the volume of a sphere is $\frac{4}{3}\pi r^3$.

Problem 230. *The ratio of the volumes of two similar spheres is 8 : 27. If the larger sphere's volume is* $135\ cm^3$*, what is the volume of the smaller solid?*

A) $90\ cm^3$	*B)* $40\ cm^3$
C) $80\ cm^3$	*D)* $50\ cm^3$

Answer: *B*
Reasoning: Since the larger sphere has a volume of 135, applying the ratio we can calculate the volume of a smaller solid by dividing by 27 and multiplying by 8.

Problem 231. *Two circles have areas of* $49\pi\ in.^2$ *and* $144\pi\ in.^2$*. What is the ratio of their radii?*

A) $49 : 144$	*B)* $49\pi : 144\pi$
C) $7 : 12$	*D)* $343 : 1728$

Answer: *C*
Reasoning: We can determine the radius of each circle by dividing by pi then square-rooting the result. The smaller circle has radius of 7 and the larger one has radius of 12.

Problem 232. *Evaluate:* $\log_5 125 =$

A) 25 B) 2
C) 3 D) 1

Answer: C
Reasoning: $5^3 = 125$

Problem 233. *Express the logarithmic equation as an exponential equation and solve:* $\log_4 \frac{1}{64} = x$
Options:
A) $x^4 = \frac{1}{64}; x = -3$
B) $4^x = \frac{1}{64}; x = -3$
C) $64^x = \frac{1}{4}; x = -3$
D) $\left(-\frac{1}{4}\right)^x = 64; x = -\frac{1}{3}$

Answer: B.
Reasoning: Definition of log.

Problem 234. *Use the fact that* $255° = 210° + 45°$ *to determine the* exact *value of* $sin 255°$.

A) $\dfrac{\sqrt{6}-\sqrt{2}}{4}$ *B)* $\dfrac{-\sqrt{2}-\sqrt{6}}{4}$
C) $-\dfrac{1}{2}$ *D)* $\dfrac{1}{2}$

Answer: B
Reasoning: From $sin(a+b) = sin(a)cos(b) + cos(a)sin(b)$, we have

$$\begin{aligned} sin(255) &= sin(210)cos(45) + cos(210)sin(45) \\ &= -\frac{1}{2} \times \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} \end{aligned}$$

Problem 235. *Find the* exact *value for* $sin 2\theta$ *given that* $sin\theta = -\frac{12}{13}$ *and* $\pi \leq \theta \leq \frac{3\pi}{2}$.

A) $\dfrac{119}{169}$ *B)* $-\dfrac{119}{169}$
C) $\dfrac{120}{169}$ *D)* $-\dfrac{120}{169}$

Answer: *C*
Reasoning:

$$sin(2x) = 2sin(x)cos(x)$$
$$= 2 \times (-12/13) \times (-5/13) \qquad = \frac{120}{169}$$

Problem 236. *Solve* $7sinx + 15 = 6sinx + 14$ *where* $0 \leq x \leq 2\pi$*.*

A) 0 | *B)* $\frac{\pi}{2}$
C) π | *D)* $\frac{3\pi}{2}$

Answer: *D*
Reasoning: $sinx = -1$, so $x = \frac{3\pi}{2}$

Problem 237. *Solve* $\cos^2\theta - 3\cos\theta - 4 = 0$ *where* $0 \leq \theta < 2\pi$*.*
Options:
A) 0
B) $\frac{\pi}{2}$
C) π
D) $\frac{3\pi}{2}$

Answer: C
Reasoning: $(\cos(\theta)-4)(\cos(\theta)+1) = 0$. Only second part can produce a solution so $\theta = \pi$.

Problem 238. *Which of the following is <u>not</u> a type of discontinuity?*

A) jump | *B) hole*
C) horizontal asymptote | *D) vertical asymptote*

Answer: *C*
Reasoning: Horizontal asymptote does not interrupt the function.

Problem 239. *Determine any points of discontinuity for* $f(x) = \frac{x(x-5)}{(x-3)(x-5)}$

A) 0 | *B)* 3
C) $3, 5$ | *D)* $0, 3, 5$

Answer: *C*
Reasoning: $x = 5$ is a hole, $x = 3$ is an infinite discontinuity.

Problem 240. *The synthetic division problem below proves which fact about* $f(x) = x^4 - 3x^3 + 7x^2 - 60x - 130$*?*

$$\begin{array}{r|rrrrr} 15 & 1 & -3 & 7 & -60 & -130 \\ & & 5 & 10 & 85 & 125 \\ \hline & 1 & 2 & 17 & 25 & -5 \end{array}$$

A) 5 is a root of f(x) *B) x-5 is a factor of f(x)*
C) $f(5) = -5$ *D)* $x^3 + 2x^2 + 17x + 25$ *is a factor of* $f(x)$

Answer: C
Reasoning: Seeing as there is a remainder of -5 when the function is divided by $x - 5$, plugging in 5 as x must result in a y of -5.

Problem 241. *Find the domain of* $f(x) = log(x - 5)$

A) $x > 0$ *B)* $x < 5$
C) $x > 5$ *D) all real numbers*

Answer: C
Reasoning: Log cannot be taken of negative numbers or zero.

Problem 242. *Identify the x and y –intercepts, if any, of the equation* $y = \frac{-1}{x+1} + 4$

A) x-int: -1, *y-int: None* *B) x-int: None, y-int:* 3
C) x-int: $-\dfrac{3}{4}$, *y-int:* 3 *D) x-int:-1, y-int:* 4

Answer: C
Reasoning: Plug in zero for x to find y intercept and do opposite for x intercept.

Problem 243. *Find the first term and the common difference of the arithmetic sequence described:* 8^{th} *term* $= 8$ *;* 20^{th} *term* $= 44$

A) $a_1 = -13; d = 3$ *B)* $a_1 = -10, d = 3$
C) $a_1 = -13, d = -3$ *D)* $a_1 = -16, d = -3$

Answer: A
Reasoning: With a 12 term difference there is a $44 - 8 = 36$ unit jump, this means the common difference is 3. Subtracting $7 \times 3 = 21$ from 8 results in a first term of -13.

Problem 244. *Which of the following is the equation of the horizontal asymptote of the graph of the function* $f(x) = \frac{4x^2}{x^3-5}$*?*
Options:
A) $x = \frac{2}{5}$
B) $x = 5$
C) $y = 0$
D) $y = 4$

Answer: C
Reasoning: Since the power in the denominator is greater than the numerator the asymptote will be 0.

Problem 245. *Simplify:* $\log_3 2 + \log_3 4 - 3\log_3 5$
Options:
A) $\log_3(-119)$
B) $\log_3\left(\frac{8}{25}\right)$
C) $\log_3\left(\frac{2}{5}\right)$
D) non-real answer

Answer: B
Reasoning: $\log_3 2 + \log_3 4 - \log_3 5^3 = \log_3 8 - \log_3 125 = \log_3\left(\frac{8}{125}\right)$

Problem 246. *Find:* $\tan^{-1}\left[\tan\left(\frac{2\pi}{3}\right)\right] \tan^{-1}\left[\tan\left(\frac{2\pi}{3}\right)\right]$

A) $\dfrac{2\pi}{3}$ *B)* $-\dfrac{\pi}{3}$
C) $\dfrac{\pi}{3}$ *D) undefined*

Answer: B
Reasoning: Since the range of tangent is only from -pi/2 to pi/2, subtract pi from 2pi/3

Problem 247. *Find:* $\sin\left[\sin^{-1}(-2)\right]$

A) 2 *B)* -2
C) $-\dfrac{1}{2}$ *D) undefined*

Answer: D
Reasoning: No inverse sin of -2.

Problem 248. $\sin^2 x - 1 =$_____

A) $\cos^2 x$ *B)* $-\cos^2 x$
C) $\csc^2 x$ *D)* $-\csc^2 x$

Answer: B
Reasoning: $\cos\hat{}\,2 + \sin\hat{}\,2 = 1$

Problem 249. $\cos^{-1}\left(\frac{1}{2}\right) =$

A) $\dfrac{\pi}{6}$ *B)* $\dfrac{\pi}{4}$
C) $\dfrac{\pi}{3}$ *D)* $\dfrac{\pi}{2}$

Answer: A
Reasoning: Unit circle.

Problem 250. *Over the interval* $[0, 2\pi)$*, solve:* $2\sin x - \sqrt{3} = 0$

A) $\dfrac{\pi}{6}$ *B)* $\dfrac{\pi}{6}, \dfrac{11\pi}{6}$
C) $\dfrac{\pi}{3}$ *D)* $\dfrac{\pi}{3}, \dfrac{11\pi}{3}$

Answer: B
Reasoning: $\sin x = \frac{\sqrt{3}}{2}$, use unit circle

Problem 251. $\tan^{-1}\left[\tan\left(\frac{5\pi}{4}\right)\right] =$

A) $\dfrac{5\pi}{4}$ *B)* $\dfrac{\pi}{4}$
C) 1 *D)* -1

Answer: B
Reasoning: Subtract pi to get within inverse tangent range.

Problem 252. *What is the exact value of* $\cos 22.5^\circ$ *?*

A) $\dfrac{\sqrt{2+\sqrt{2}}}{2}$ *B)* $-\dfrac{\sqrt{2+\sqrt{2}}}{2}$
C) $\dfrac{\sqrt{2-\sqrt{2}}}{2}$ *D)* $-\dfrac{\sqrt{2-\sqrt{2}}}{2}$

Answer: A
Reasoning: $\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}, \cos 22.5 = \sqrt{\frac{1+\cos 45}{2}}$

change $\sin^2 x$ to $1 - \cos^2 x$ then factor for $(\cos x - 2)(\cos x - 1) = 0$, only right part can be solved for.

Problem 253. *Which one is a solution to the equation:* $\sqrt{3}\tan x + 1 = 0$

A) $\frac{-\pi}{3}$ *B)* $\frac{\pi}{6}$

C) $\frac{5\pi}{6}$ *D)* $\frac{2\pi}{3}$

Answer: C
Reasoning: $\tan x = -\frac{1}{\sqrt{3}}$

Problem 254. *Calculate the coefficient of* x^2 *in the expansion of* $(x-3)^5$

A) 270 *B)* 90

C) -17 *D)* -270

Answer: D
Reasoning: Each factor with a coefficient of x^2 will be multiplied by -3 3 times. This means each factor has a coefficient of -27. There will be 10 such factors.

Problem 255. *What is the expansion of the polynomial* $(x-2)^4$ *?*

A) $x^4 + 16$ *B)* $x^4 - 8x^3 + 24x^2 - 32x + 16$

C) $x^4 - 16x^3 + 32x^2 - 32x + 16$ *D)* $x^4 + 4x^3 + 6x^2 + 4x + 1$

Answer: B
Reasoning: Use polynomial expansion formula.

Problem 256. *Find the sum.* $\sum_{n=1}^{10} 4n - 5$
Options:
A) 235
B) 35
C) 36
D) 170

Answer: D
Reasoning: Since this is an arithmetic sequence we can sum the first and last term then multiply by the amount of terms and divide by 2 for the sum. $(-1+35) \times \frac{10}{2}$

Problem 257. *Find the sum.* $\sum_{k=1}^{\infty} 6\left(-\frac{2}{3}\right)^{k-1}$
Options:
A) 0.6
B) -0.6
C) 3.6
D) -3.6

Answer: C
Reasoning: Since this is an infinite geometric sequence: $\frac{a}{1-r} = \frac{6}{1+\frac{2}{3}} = \frac{18}{5} = 3.6$

Problem 258. *Given* $\triangle ABC$*, where* $\angle \mathrm{A} = 41°, \angle \mathrm{B} = 58°$*, and* $\mathrm{c} = 19.7$ cm*, determine the measure of side* b*.*

A) *not possible*	*B)* 16.91 cm
C) 0.89 cm	*D)* 12.94 cm

Answer: B
Reasoning: Using law of sines $\frac{\sin 58}{x} = \frac{\sin 81}{19.7}$

Problem 259. *Given* $\triangle ABC$*, where* $\mathrm{a} = 9,\ \mathrm{b} = 12,$ *and* $\mathrm{c} = 16$*, determine the measure of angle* B*. Round to the nearest tenth.*

A) *not possible*	*B)* 132.1°
C) 47.9°	*D)* 1°

Answer: C
Reasoning: Use law of cosines

Problem 260. *In* $\triangle ABC, A = 47°, B = 56°,$ *and* $c = 14$*, find* b*.*

A) 77	*B)* 7.9
C) 10.5	*D)* 11.9

Answer: D
Reasoning: Use law of sines

Problem 261. *Evaluate* $\tan(\alpha - \beta)$ *given:* $\tan\alpha = -\frac{4}{3}, \frac{\pi}{2} < \alpha < \pi$ *and* $\cos\beta = \frac{1}{2}, 0 < \beta < \frac{\pi}{2}$*.*
Options:
A) $\frac{25\sqrt{3}+48}{39}$
B) $-\frac{25\sqrt{3}+48}{39}$
C) $\frac{16+7\sqrt{3}}{47}$
D) $-\frac{16+7\sqrt{3}}{47}$

Answer: A
Reasoning: $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{-\frac{4}{3}-\sqrt{3}}{1-\frac{4}{3}\sqrt{3}}$

Problem 262. *Evaluate* $p(x) = x^3 + x^2 - 11x + 12$ *for* $x = 2$*.*

Answer: 2
Reasoning: Plug in 2.

Problem 263. *Write* $\ln \frac{x^2(y^2-z)^3}{\sqrt{y+1}}$ *as the sum and/or difference of logarithms. Express powers as factors.*

Answer: $2\ln x + 3\ln(y^2 - z) - \frac{1}{2}\ln(y+1)$
Reasoning: Split using log rules $2\ln x + 3\ln(y^2 - z) - \frac{1}{2}\ln(y+1)$.

Problem 264. *Rewrite the following as the log of a single expression and simplify.*

$$\frac{1}{3}\log 125 + 2\log(x-1) - 3\log(x+4)$$

Answer: $\log(5 \times (x-7)^2/(x+4)^3)$
Reasoning: first move all powers then combine: $\log 125^{1/3} + \log(x-1)^2 - \log(x+4)^3$

$$\log\left(5 * (x-7)^2/(x+4)^3\right)$$

Problem 265. *Solve* $27^{3x} = 81$ *for* x.

Answer: 4/9
Reasoning: $27^{\frac{4}{3}} = 81, 3x = \frac{4}{3}, x = \frac{4}{9}$

Problem 266. *Use long division to divide* $f(x) = 6x^3 - x^2 - 5x + 2$ *by* $3x - 2$.

Answer: $2x^2 + x - 1$
Reasoning: $\frac{6x^3-x^2-5x+2}{3x-2} = 2x^2 + \frac{3x^2-5x+2}{3x-2} = 2x^2 + x + \frac{-3x+2}{3x-2} = 2x^2 + x - 1$

Problem 267. *A culture of bacteria obeys the law of uninhibited growth. If 500 bacteria are present initially and there are 800 after 1 hour, how many will be present after 5 hours?*

Answer: 5242.88
Reasoning: The rate of growth is 800/500 per hour so $500 \times \frac{8^5}{5} = 5242.88$

Problem 268. *If* $f(x)$ *is the function given by* $f(x) = e^{3x} + 1$, *at what value of* x *is the slope of the tangent line to* $f(x)$ *equal to 2?*
Options:
A) -0.173
B) 0
C) -0.135
D) -0.366
E) 0.231

Answer: C
Reasoning: $f'(x) = 3e^{3x} = 2.\ \ln(2) = 3x, x = -0.135$

Problem 269. *Which of the following is an equation for a line tangent to the graph of* $f(x) = e^{3x}$ *when* $f'(x) = 9$*?*
Options:
A) $y = 3x + 2.633$
B) $y = 9x - 0.366$
C) $y = 9x - 0.295$
D) $y = 3x - 0.295$
E) None of these

Answer: C
Reasoning: $f'(x) = 3e^{3x} = 9, x = 0.336, y - 3 = 9(x - 0.336)$

Problem 270. *If $f'(x) = \ln x - x + 2$, at which of the following values of x does f have a relative maximum value?*

A) 3.146 *B)* 0.159
C) 1.000 *D)* 4.505
E) None of these

Answer: A
Reasoning: Check when function crosses x axis by graphing

Problem 271. $\int \frac{4x}{16+x^4}\, dx =$
Options:
A) $\frac{1}{4}\sec^{-1}\frac{x^2}{4} + C$
B) $\frac{1}{2}\tan^{-1}\frac{x^2}{4} + C$
C) $\frac{1}{8}\sec^{-1}\frac{x^2}{4} + C$
D) $2\tan^{-1}\frac{x^2}{4} + C$
E) None of these

Answer: B
Reasoning: Use inverse tangent derivatives

Problem 272. *If $f(x) = 3x^2 - x$, and $g(x) = f^{-1}(x)$ over the domain $[0, \infty)$, then $g'(10)$ could be which of the following?*

A) 59 *B)* $\frac{1}{59}$
C) $\frac{1}{10}$ *D)* 11
E) $\frac{1}{11}$

Answer: E
Reasoning: $10 = 3x^2 - x, (3x + 5)(x - 2) = 0, x = 2, f'(2) = 11, g'(10) = \frac{1}{11}$

Problem 273. *Find the distance traveled in the first four seconds for a particle whose velocity is given by $v(t) = 7e^{-t^2}$, where t stands for time.*

A) 0.976 *B)* 6.204
C) 6.359 *D)* 12.720
E) 7.000

Answer: B
Reasoning: Take the integral of velocity for distance.

Problem 274. *Find* $\lim_{x\to 0} -\frac{\sin(5x)}{\sin(4x)}$

A) 0 *B)* 1
C) $-5/4$ *D)* $5/4$
E) None of these

Answer: C
Reasoning: Use lhopitals rule to take derivative of numerator and denominator of limit.

Problem 275. *Find the area* R *bounded by the graphs of* $y = \sqrt{x}$ *and* $y = x^2$

A) 0.333 *B)* -0.333
C) 1.000 *D)* -1.000
E) None of these

Answer: A
Reasoning: $\int_0^1 \sqrt{\chi} - x^2 dx = 0.333$

Problem 276.

$$\frac{d}{dx}\int_0^{3x} \cos(t)\,dt =$$

Options:
A) $\sin 3x$
B) $-\sin 3x$
C) $\cos 3x$
D) $3\sin 3x$
E) $3\cos 3x$

Answer: E
Reasoning: Integrate to sin then chain rule out a 3.

Problem 277. *The average value of the function* $f(x) = (x-1)^2$ *on the interval [1,5] is:*

A) $-\dfrac{16}{3}$ *B)* $\dfrac{16}{3}$
C) $\dfrac{64}{5}$ *D)* $\dfrac{66}{3}$
E) $\dfrac{256}{3}$

Answer: B
Reasoning: $\frac{\int_7^5 (x-7)^2 dx}{4} = \frac{16}{3}$

Problem 278. *Write the following expression as a logarithm of a single quantity:* $\ln x - 12\ln(x^2-1)$

A) $\ln\left(\dfrac{x}{(x^2-1)^{-12}}\right)$ *B)* $\ln\left(\dfrac{x}{12(x^2-1)}\right)$
C) $\ln\left(x - 12\left(x^2-1\right)\right)$ *D)* $\ln\left(\dfrac{x}{(x^2-1)^{12}}\right)$
E) None of these

Answer: D
Reasoning: Use log rules to turn 12 into an exponent and change the subtraction into one log division

Problem 279. *Find an equation of the tangent line to the graph of* $y = \ln(x^2)$ *at the point* $(1, 0)$.

A) $y = x - 2$ *B)* $\mathrm{y} = 2(x + 1)$
C) $y = 2(x - 1)$ *D)* $y = x - 1$
E) None of these

Answer: C
Reasoning: $y' = \frac{2x}{x^2} = \frac{2}{x}, y'(7) = 2$

Problem 280. *Find the area* R *bounded by the graphs of* $y = x$ *and* $y = x^2$

A) $\dfrac{1}{6}$ *B)* $\dfrac{1}{2}$
C) $\dfrac{-1}{6}$ *D)* $\dfrac{-1}{2}$
E) None of these

Answer: A
Reasoning: $\int_0^1 x - x^2 dx$

Problem 281. *Find the indefinite integral:* $\int \frac{x}{-2x^2+3} dx$

A) $\dfrac{1}{-4x} + C$ *B)* $\ln\left|-2x^2 + 3\right| + C$
C) $\dfrac{-1}{4} \ln\left|-2x^2 + 3\right| + C$ *D)* $\dfrac{\ln|-2x^2 + 3|}{-2x^2 + 3} + C$
E) None of these

Answer: C
Reasoning: u substitution with $u = -2x^2 + 3$

Problem 282. *Find the indefinite integral:* $\int x \ln(x) dx$

A) $\dfrac{(\ln x)^2}{x} + C$ *B)* $\dfrac{x^2 \ln(x)}{2} - \dfrac{x^2}{4} + C$
C) $x \ln(x) + C$ *D)* $\ln(x) + 1 + C$
E) None of these

Answer: B
Reasoning: Integration by parts.

Problem 283. *Using the substitution* $u = 2x + 1$, $\int_0^2 \sqrt{2x+1}dx$ *is equivalent to which of the following?*

A) $\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{u}$ B) $\frac{1}{2}\int_0^2 \sqrt{u}du$

C) $\frac{1}{2}\int_1^5 \sqrt{u}du$ D) $\int_0^2 \sqrt{u}du$

E) None of these

Answer: C
Reasoning: Find new limits by plugging in 0 and 2 to the equation to get 1 and 5.

Problem 284. *Region R is the area bounded by the graphs of* $y = x$ *and* $y = x^3$. *Find the volume of the solid generated when R is revolved about the x-axis.*

A) $\frac{\pi}{3}$ B) $\frac{21\pi}{4}$

C) $\frac{4\pi}{21}$ D) 3π

E) None of these

Answer: C
Reasoning: $\int_0^1 \Pi\left(x^2 - x^6\right) dx$

Problem 285. *Find the indefinite integral:* $\int xe^{2x}dx$

A) $\frac{e^{2x}}{x} + \frac{x}{e^{2x}} + C$ B) $\frac{\ln(x)}{e} + C$

C) $\frac{x}{e^{2x}} + C$ D) $\frac{xe^{2x}}{2x} - \frac{e^{2x}}{4} + C$

E) None of these

Answer: E
Reasoning: Integration by parts

Problem 286. $\int x\sqrt{x+3}dx =$

A) $\frac{2}{3}x^{\frac{3}{2}} + 6x^{\frac{1}{2}} + C$ B) $\frac{2(x+3)^{\frac{3}{2}}}{3} + C$

C) $\frac{3(x+3)^{\frac{3}{2}}}{2} + C$ D) $\frac{4x^2(x+3)^{\frac{3}{2}}}{3} + C$

E) $\frac{2}{5}(x+3)^{\frac{5}{2}} - 2(x+3)^{\frac{3}{2}} + C$

Answer: E
Reasoning: Integration by parts.

Problem 287. *Consider the differential equation* $\frac{dy}{dx} = \frac{y-1}{x^3}$*, where* $x \neq 0$*. Find the general solution* $y = f(x)$ *to the differential equation.*

Answer:$y = ce^{-\frac{1}{x}} + 1$.
Reasoning: $\frac{1}{y-1}dy = \frac{1}{x^2}dx, \ln(y-1) = -x^{-1} + c, y = ce^{-\frac{1}{x}} + 1$.

Problem 288. *Compute the determinant of the matrix*

$$B = \begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}.$$

Answer: 10.
Solution: Using the definition of the determinant,

$$\det(B) = 3(0 \cdot 1 - (-2) \cdot 1) - 0 + 2(2 \cdot 1 - 0 \cdot (-2)) = 10.$$

Problem 289. *Let*

$$A = \begin{pmatrix} a & 0 & c & b \\ 1 & 0 & 1 & 3 \\ 2 & 1 & -1 & 4 \\ 0 & 1 & 1 & 5 \end{pmatrix}.$$

and A_{ij} *be the algebraic cofactors of* A*. Compute* $A_{11} + A_{12} + A_{13} + A_{14}$.

Answer: 21.
Solution:

$$A_{11} + A_{12} + A_{13} + A_{14} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 3 \\ 2 & 1 & -1 & 4 \\ 0 & 1 & 1 & 5 \end{vmatrix} = 21.$$

Problem 290. *Find the solution* $[x_1, x_2, x_3]$ *to the following equations*

$$\begin{cases} x_1 + 3x_2 + 3x_3 = 16, \\ 3x_1 + x_2 + 3x_3 = 14, \\ 3x_1 + 3x_2 + x_3 = 12. \end{cases}$$

Answer: [1,2,3].
Solution: The second equation subtracts the first one, leading to

$$x_1 - x_2 = -1.$$

The third equation subtracts the second one, leading to

$$x_2 - x_3 = -1.$$

Thus

$$x_2 = x_1 + 1, \quad x_3 = x_1 + 2.$$

Inserting x_2, x_3 into the first one, we deduce that

$$7x_1 + 9 = 16.$$

Then $x_1 = 1, x_2 = 2, x_3 = 3$ is the solution.

Problem 291. *Find the positively definite matrix* $A \in \mathbb{R}^{3\times 3}$ *such that*

$$A^2 = \begin{pmatrix} 11 & 7 & 7 \\ 7 & 11 & 7 \\ 7 & 7 & 11 \end{pmatrix}.$$

In your answer, present the matrix in the form of $[a_{11}, a_{12}, a_{13}; a_{21}, a_{22}, a_{23}; a_{31}, a_{32}, a_{33}]$

Answer: [3, 1, 1; 1, 3, 1; 1, 1, 3]
Solution: Let

$$B = \begin{pmatrix} 11 & 7 & 7 \\ 7 & 11 & 7 \\ 7 & 7 & 11 \end{pmatrix}.$$

The characteristic polynomial of B is

$$\begin{vmatrix} \lambda - 11 & -7 & -7 \\ -7 & \lambda - 11 & -7 \\ -7 & -7 & \lambda - 11 \end{vmatrix} = (\lambda - 25)(\lambda - 4)^2.$$

Thus, the eigenvalues of A are $25, 4, 4$. The corresponding eigenvectors are $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},$ $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Set

$$U = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \end{pmatrix},$$

then

$$B = U \begin{pmatrix} 25 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} U^{-1}.$$

Thus

$$A = U \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} U^{-1} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

Problem 292. *Compute the volume of the triangular pyramid generated by four points* $(1, 1, 1), (2, 5, 5),$ $(5, 2, 5),$ *and* $(5, 5, 2)$ *in* $\mathbb{R}^3$.

Answer: 13.5

Solution: Using the geometry meaning of the determinant, we know the volume of the triangular pyramid can be represented as

$$\frac{1}{6}\begin{vmatrix} 1 & 4 & 4 \\ 4 & 1 & 4 \\ 4 & 4 & 1 \end{vmatrix} = \frac{27}{2}.$$

Problem 293. *Find the values of* $[a, b]$ *such that* $(1, 2, 1)^\top$ *is an eigenvector of the matrix* $\begin{pmatrix} 1 & 2 & 1 \\ 3 & a & b \\ a & 0 & b \end{pmatrix}$. *Present the answer as* $[a, b]$.

Answer: [3,3]
Solution: Let λ be the eigenvalue corresponding to eigenvector $(1, 2, 1)^\top$, then

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & a & b \\ a & 0 & b \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

From this, we deduce that

$$6 = \lambda, 3 + 2a + b = 2\lambda, a + b = \lambda.$$

Solving the equation, we have $a = b = 3$.

Problem 294. *Find the matrix* A *whose eigenvalues are 2,3,6 and corresponding eigenvectors are* $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ *respectively.*
In your answer, present the matrix in the form of $[a_{11}, a_{12}, a_{13}; a_{21}, a_{22}, a_{23}; a_{31}, a_{32}, a_{33}]$.

Answer: [3, -1, 1; -1, 5, -1; 1, -1, 3].
Solution: Let

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

then

$$AU = U\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

Note that U is an orthogonal matrix, then $U^{-1} = U^\top$. Thus

$$A = U\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}U^\top = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}.$$

Problem 295. *Compute the rank of the matrix*

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \\ 3 & 1 & 4 \end{pmatrix}$$

Answer: 2
Solution: By elementary transformation of matrix, we have

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \\ 3 & 1 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 3 & 1 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & -2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

It implies that the rank is 2.

Problem 296 (Rank of a matrix)**.** *Compute the dimension of the linear subspace generated by the following vectors*

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 5 \\ 5 \end{pmatrix}.$$

Answer: 3.
Solution: Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & -1 & 2 \\ 1 & 1 & 3 & 5 \\ 1 & 0 & 4 & 5 \end{pmatrix}$$

Then the dimension of the linear subspace generated by the column vectors of matrix A is rank(A). By elementary transformation of matrix, we have

$$A \to \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & -1 & 4 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 3 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It shows that the rank of A is 3. Thus, the dimension is 3.

Problem 297. *Let the matrix* $A = \begin{pmatrix} 2 & -2 & 1 \\ 4 & -4 & 2 \\ 6 & -6 & 3 \end{pmatrix}$. *Compute the product matrix* A^{2024}.

In your answer, present the matrix in the form of $[a_{11}, a_{12}, a_{13}; a_{21}, a_{22}, a_{23}; a_{31}, a_{32}, a_{33}]$.

Answer: [2, -2, 1; 4, -4, 2; 6, -6, 3].

Solution: Note that $A^2 = A$, so $A^{2024} = A = \begin{pmatrix} 2 & -2 & 1 \\ 4 & -4 & 2 \\ 6 & -6 & 3 \end{pmatrix}$.

Problem 298. *Compute* $|A^{-1}|$ *for* $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$.

Answer: 1.
Solution: Since $AA^{-1} = I_3$ and $|AA^{-1}| = |A||A^{-1}|$, we then obtain

$$|A^{-1}| = \frac{1}{|A|} = 1.$$

Problem 299. *Compute* $|A^*|$ *for* $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, *where* A^* *is the adjoint matrix of* A.

Answer: 4.
Solution: Since $AA^* = |A|I_3$ and $|A| = 2$, we then obtain

$$|A^*| = |A|^2 = 4.$$

Problem 300. *Suppose that* $A \in R^{3\times 3}$ *is a matrix with* $|A| = 1$, *compute* $|A^* - 2A^{-1}|$, *where* A^* *is the adjoint matrix of* A.

Answer: -1.
Solution: Note the identity $AA^* = |A|I_3$ and $|A| = 1$, we know that

$$A^* = A^{-1}.$$

Thus

$$|A^* - 2A^{-1}| = |-A^{-1}| = (-1)^3|A^{-1}| = -1 \cdot \frac{1}{|A|} = 1.$$

Problem 301. *Let* A^* *denote the adjoint matrix of matrix* A. *Suppose that* $A^* = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$, *and the determinant is* $|A| = 1$, *Find* A.

In your answer, present the matrix in the form of $[a_{11}, a_{12}, a_{13}; a_{21}, a_{22}, a_{23}; a_{31}, a_{32}, a_{33}]$.

Answer: [1, -2, 5; 0, 1, -4; 0, 0, 1].

Solution: It follows from the equation $AA^* = |A|I_3$ that

$$A = |A|(A^*)^{-1}.$$

By the assumption $|A| = 1$, we have $A = (A^*)^{-1}$. By the formula

$$(A^*)^{-1} = \frac{1}{|A^*|}(A^*)^*.$$

By the definition of adjoint matrixes, we have

$$(A^*)^* = \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have $|A^*| = 1$ by a direct computation. Consequently, $A = \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$.

Problem 302. *Suppose that the vectors* $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ *and vectors* $\begin{pmatrix} 0 \\ a \\ -1 \end{pmatrix}$, $\begin{pmatrix} b \\ 3 \\ 1 \end{pmatrix}$ *generated the same linear subspace. Compute a and b. Present the answer as $[a, b]$.*

Answer: [1,2]
Solution: The two sets of vectors can be linearly represented by each other. By elementary transformation, we have

$$\begin{pmatrix} 1 & 1 & 0 & 0 & b \\ 1 & 2 & 1 & a & 3 \\ 1 & 0 & -1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 0 & b \\ 0 & 1 & 1 & a & 3-b \\ 0 & -1 & -1 & -1 & 1-b \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 0 & b \\ 0 & 1 & 1 & a & 3-b \\ 0 & 0 & 0 & a-1 & 4-2b \end{pmatrix}$$

Thus $a - 1 = 4 - 2b = 0$. It implies that $a = 1, b = 2$.

Problem 303. *Suppose that $A = \begin{pmatrix} 1 & 2 \\ 2 & a \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ are similar matrixes, find a and b. Present the answer in the form of $[a, b]$.*

Answer: [4,5]

Solution: Since A and B are similar matrixes, then

$$|A| = |B|, \quad \mathrm{tr}(A) = \mathrm{tr}(B).$$

It shows that

$$a - 4 = 0, \quad 1 + a = 0 + b.$$

Thus $a = 4, b = 5$.

Problem 304. *Suppose there are two matrixes $A \in \mathbb{R}^{3\times 4}, B \in \mathbb{R}^{4\times 3}$ satisfying that*

$$AB = \begin{pmatrix} -9 & 2 & 2 \\ -20 & 5 & 4 \\ -35 & 7 & 8 \end{pmatrix}, \quad BA = \begin{pmatrix} -14 & 2a-5 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ -15 & 3a-3 & 3 & 6 \\ -32 & 6a-7 & 4 & 14 \end{pmatrix}.$$

Compute a.

Answer: -2

Solution: By the identity

$$3 - \mathrm{rank}(I_3 - AB) = 4 - \mathrm{rank}(I_4 - BA),$$

and note that

$$\operatorname{rank}(I_3 - AB) = 1,$$

It implies that

$$\operatorname{rank}(I_4 - BA) = 2.$$

Since

$$I_4 - BA = \begin{pmatrix} 15 & 5-2a & -2 & -6 \\ 0 & 0 & 0 & 0 \\ 15 & 3-3a & -2 & -6 \\ 32 & 7-6a & -4 & -13 \end{pmatrix}.$$

It indicates that

$$\begin{vmatrix} 5-2a & -2 & -6 \\ 3-3a & -2 & -6 \\ 7-6a & -4 & -13 \end{vmatrix} = 0.$$

Thus $a = -2$.

Problem 305. *Suppose that $A \in \mathbb{R}^{3\times 2}, B \in \mathbb{R}^{2\times 3}$ satisfy*

$$AB = \begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix},$$

Compute BA. Present the matrix in the form of $[a_{11}, a_{12}; a_{21}, a_{22}]$.

Answer: [9,0; 0, 9]
Solution: By the identity

$$3 - \operatorname{rank}(9I_3 - AB) = 2 - \operatorname{rank}(9I_2 - BA),$$

and note that

$$\operatorname{rank}(9I_3 - AB) = \operatorname{rank}\begin{pmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{pmatrix} = 1,$$

it implies that $\operatorname{rank}(9I_2 - BA) = 0$. Thus

$$BA = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}.$$

Problem 306. *Compute a, b, c such that the linear equations*

$$\begin{cases} -2x_1 + x_2 + ax_3 - 5x_4 = 1, \\ x_1 + x_2 - x_3 + bx_4 = 4, \\ 3x_1 + x_2 + x_3 + 2x_4 = c \end{cases}$$

and the linear equations

$$\begin{cases} x_1 + x_4 = 1, \\ x_2 - 2x_4 = 2, \\ x_3 + x_4 = -1. \end{cases}$$

have the same set of solutions. Present the answer as $[a, b, c]$.

Answer: [-1,-2,4]
Solution: The general solutio to the equation

$$\begin{cases} x_1 + x_4 = 1, \\ x_2 - 2x_4 = 2, \\ x_3 + x_4 = -1. \end{cases}$$

can be written as

$$x_1 = 1 - x_4, x_2 = 2 + 2x_4, x_3 = -1 - x_4, \quad x_4 \in \mathbb{R}.$$

Inserting them into the first equation, we obtain that

$$\begin{cases} (-1 - a)x_4 = 1 + a, \\ (2 + b)x_4 = 0, \\ c = 4. \end{cases}$$

Since x_4 is an arbitrary constant, we deduce that $a = -1, b = -2, c = 4$.

Problem 307. *Suppose that $\phi : \mathbb{R}^{3\times 3} \to \mathbb{R}$ is a mapping which satisfies the following properties*

1. *$\phi(AB) = \phi(A)\phi(B)$ for any $A, B \in \mathbb{R}^N$. and*
2. *$\phi(A) = |A|$ for any diagonal matrix A.*

Compute $\phi(A)$ for

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Answer: 4
Solution: Note that A is symmetric, so there exists an invertible matrix P such that

$$A = P\text{diag}(\lambda_1, \lambda_2, \lambda_3)P^{-1}.$$

By the first property of ϕ, we have

$$\phi(A) = \phi(P)\phi(\text{diag}(\lambda_1, \lambda_2, \lambda_3))\phi(P^{-1}).$$

Also we know

$$\phi(P)\phi(P^{-1}) = \phi(PP^{-1}) = \phi(I_3) = |I_3| = 1$$

due to the second property. Thus

$$\phi(A) = \phi(\text{diag}(\lambda_1, \lambda_2, \lambda_3)) = \lambda_1\lambda_2\lambda_3 = |A| = 4.$$

Problem 308. *Suppose that $\psi : \mathbb{R}^{3\times 3} \to \mathbb{R}$ is a mapping which satisfies the following properties*

1. *$\psi(AB) = \psi(BA)$ for any $A, B \in \mathbb{R}^N$. and*

2. $\psi(A) = \operatorname{tr}(A)$ *for any diagonal matrix* A.

Compute $\psi(A)$ *for*

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}.$$

Answer: 3
Solution: Note that A is symmetric, so there exists an invertible matrix P such that

$$A = P\operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)P^{-1}.$$

By the first property of ψ, we have

$$\psi(A) = \psi(\operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)P^{-1}P) = \psi(\operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)).$$

Also we know

$$\psi(\operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)) = \lambda_1 + \lambda_2 + \lambda_3.$$

due to the second property. Thus

$$\psi(A) = \lambda_1 + \lambda_2 + \lambda_3 = \operatorname{tr}(A) = 3.$$

Problem 309. *Compute the limit* $\lim_{n\to\infty} \frac{y_n}{x_n}$, *where the two sequence* $\{x_n\}, \{y_n\}$ *are defined by*

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

with $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

Answer: 1.62

Solution: The characteristic polynomial of A is

$$\begin{vmatrix} \lambda & -1 \\ -1 & \lambda - 1 \end{vmatrix} = \lambda^2 - \lambda - 1.$$

Thus the eigenvalues are $\lambda_1 = \frac{1+\sqrt{5}}{2}, \lambda_2 = \frac{1-\sqrt{5}}{2}$. Their eigenvectors are $\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$ respectively. Set

$$P = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix},$$

then

$$A = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}.$$

Thus

$$A^n = P \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} P^{-1}.$$

Since

$$P^{-1} = \frac{-1}{\sqrt{5}} \begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix}$$

we have

$$A^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} \\ \lambda_1^{n+2} - \lambda_2^{n+2} \end{pmatrix}.$$

Therefore $x_n = \frac{1}{\sqrt{5}}\left(\lambda_1^{n+1} - \lambda_2^{n+1}\right)$ and $y_n = \frac{1}{\sqrt{5}}\left(\lambda_1^{n+2} - \lambda_2^{n+2}\right)$.

Then, we obtain that

$$\lim_{n\to\infty} \frac{y_n}{x_n} = \lambda_1 = \frac{1+\sqrt{5}}{2}.$$

Problem 310. *Find the integer a such that $x^2 - x + a$ is a factor of $x^{13} + x + 90$.*

Answer: 2
Solution: Let $x^{13}+x+90 = (x^2-x+a)q(x)$, where $q(x) \in \mathbb{Z}[x]$ is a polynomial with integral coefficients. Inserting $x = 0, 1$ into $x^{13}+x+90 = (x^2-x+a)q(x)$ leads to $a|90, a|92$. Namely a is a factor of 90 and 92. Thus $a|2$. Then $a = 1, -1, 2$ or -2. Note that $x^{13}+x+90 = 0$ has no positive root, therefore $a = 1$ or 2. Again inserting $x = -1$ into $x^{13}+x+90 = (x^2-x+a)q(x)$, we obtain $(a+2)|88$. Then $a = 2$. Indeed,

$$x^{13} + x + 90x^2 - x + 2$$

Problem 311. *Find the integer coefficient polynomial with the smallest degree that has a root $\sqrt{2} + \sqrt{3}$.*

Answer: $x^4 - 10x^2 + 1$.
Solution: Since $\sqrt{2} + \sqrt{3}$ is a root, its conjugates $\pm\sqrt{2} \pm \sqrt{3}$ are also possible roots since the coefficients are integers. Let

$$f(x) = (x - \sqrt{2} - \sqrt{3})(x - \sqrt{2} + \sqrt{3})(x + \sqrt{2} - \sqrt{3})(x + \sqrt{2} + \sqrt{3}),$$

that is, $f(x) = x^4 - 10x^2 + 1$. Suppose that $g(x)$ is the desired polynomial. Then $g(x)|f(x)$. Therefore, there exists an integer coefficient polynomial $h(x)$ such that

$$f(x) = g(x)h(x).$$

On the one hand, the degree of $g(x)$ is not 1 because $x - \sqrt{2} - \sqrt{3}$ does not have integer coefficients. On the other hand, the degrees of $g(x)$ cannot be two because otherwise, the coefficient of x is not an integer when the roots are two of $\pm\sqrt{2} \pm \sqrt{3}$. Similarly, the degree of g cannot be three. Consequently, $g(x) = f(x) = x^4 - 10x^2 + 1$ is the desired polynomial.

Problem 312. *Let $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix}$ and $v = (2, 1, 0)^\top$, find the polynomial $f(x)$ with the least degree such that $f(A)v = 0$.*

Answer: $x^2 - 8x + 7$.
Solution: By a direct calculation, we obtain the characteristic polynomial of A is

$$(\lambda - 7)(\lambda - 1)^2.$$

So $f(x)$ must be one of the five factors $x-1$, $x-7$, $(x-1)^2$, $(x-1)(x-7)$ and $(x-1)^2(x-7)$.
Note that

$$Av = \begin{pmatrix} 8 \\ 7 \\ 6 \end{pmatrix}$$

thus $f(x)$ is neither $x - 1$ nor $x - 7$. Since

$$(A - I_3)v = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}$$

and

$$(A - 7I_3) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

therefore

$$(A - 7I_3)(A - I_3)v = 0.$$

Then we deduce that $f(x) = (x - 7)(x - 1) = x^2 - 8x + 7$.

Problem 313. *Evaluate the following limit:*

$$\lim_{n\to\infty}\left(\sqrt{n^2+2n-1}-\sqrt{n^2+3}\right).$$

Answer: 1.
Solution:

$$\begin{aligned}
\lim_{n\to\infty}\left(\sqrt{n^2+2n-1}-\sqrt{n^2+3}\right) &= \lim_{n\to\infty}\left(\sqrt{n^2+2n-1}-\sqrt{n^2+3}\right)\cdot\frac{\sqrt{n^2+2n-1}+\sqrt{n^2+3}}{\sqrt{n^2+2n-1}+\sqrt{n^2+3}}\\
&= \lim_{n\to\infty}\frac{(n^2+2n-1)-(n^2+3)}{\sqrt{n^2+2n-1}+\sqrt{n^2+3}}\\
&= \lim_{n\to\infty}\frac{2n-4}{\sqrt{n^2+2n-1}+\sqrt{n^2+3}}\\
&= \lim_{n\to\infty}\frac{\frac{1}{n}(2n-4)}{\frac{1}{n}\left(\sqrt{n^2+2n-1}+\sqrt{n^2+3}\right)}\\
&= \lim_{n\to\infty}\frac{2-\frac{4}{n}}{\sqrt{1+\frac{2}{n}-\frac{1}{n^2}}+\sqrt{1+\frac{3}{n}}}\\
&= \frac{2-0}{\sqrt{1+0-0}+\sqrt{1+0}}\\
&= 1.
\end{aligned}$$

Problem 314. *Find the limit*

$$\lim_{x\to 1}\frac{f(2x^2+x-3)-f(0)}{x-1}$$

given $f'(1)=2$ *and* $f'(0)=-1$.

Answer: -5.
Solution: Let $g(x)=2x^2+x-3$. Noticing that $g(1)=0$, the desired limit equals $\lim_{x\to 1}\frac{f(g(x))-f(g(1))}{x-1}$. By the definition of the derivative and the chain rule and noting that $g'(1)=5$, we have

$$\lim_{x\to 1}\frac{f(g(x))-f(g(1))}{x-1}=f'(g(1))g'(1)=f'(0)g'(1)=(-1)(5)=-5.$$

Problem 315. *Evaluate* $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}$.

Answer: 4
Solution:

$$\begin{aligned}\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2} &= \lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2}\\ &= \lim_{x\to 4}\frac{(x-4)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{x}+2)}\\ &= \lim_{x\to 4}\frac{(x-4)(\sqrt{x}+2)}{x-4} = \lim_{x\to 4}(\sqrt{x}+2) = 4.\end{aligned}$$

Problem 316. *Find the values of a such that the function $f(x)$ is continuous on $\mathbb{R}$, where $f(x)$ is defined as*

$$f(x) = \begin{cases} 2x-1, & \text{if } x \le 0,\\ a(x-1)^2-3, & \text{otherwise.}\end{cases}$$

Answer: 2.
Solution: By the definition of $f(x)$, we have

$$\begin{aligned} f(0) &= -1;\\ \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-}(2x-1) = 2(0)-1 = -1;\\ \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+}(a(x-1)^2-3) = a(0-1)^2-3 = a-3.\end{aligned}$$

To obtain the continuity of $f(x)$ at $x=0$, we need $-1 = a-3$, that is, $a=2$.
So, the function $f(x)$ is continuous at $x=0$ when $a=2$.

Problem 317. *Evaluate* $\lim\limits_{x\to 1}\frac{x^2-1}{x+1}$.

Answer: 0
Solution: Use direct substitution to obtain the result:

$$\lim_{x\to 1}\frac{x^2-1}{x+1} = \frac{1^2-1}{1+1} = \frac{0}{2} = 0.$$

Problem 318. *Evaluate the integral* $\displaystyle\int_1^e \ln x\; dx$.

Answer: 1
Solution: Use integration by parts:

$$\int u\,dv = uv - \int v\,du.$$

Choose $u = \ln x$ and $dv = dx$, then $du = \frac{1}{x}\,dx, v = x$. Apply the integration by parts formula:

$$\int_1^e \ln x\,dx = x\ln x\Big|_1^e - \int_1^e x\left(\frac{1}{x}\right)\,dx = (e-0)-(e-1) = 1.$$

Problem 319. *Let $f(3) = -1$, $f'(3) = 0$, $g(3) = 2$ and $g'(3) = 5$. Evaluate $\left(\frac{f}{g}\right)'(3)$.*

Answer: 1.25

Solution: Use the quotient rule. The quotient rule gives

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

Now, using that $f(3) = -1$, $f'(3) = 0$, $g(3) = 2$, and $g'(3) = 5$, we have

$$\left(\frac{f}{g}\right)'(3) = \frac{f'(3)g(3) - f(3)g'(3)}{g(3)^2} = \frac{0\cdot 2 - (-1)\cdot 5}{2^2} = \frac{5}{4}.$$

Problem 320. *Find all value(s) of x at which the tangent line(s) to the graph of $y = -x^2 + 2x - 3$ are perpendicular to the line $y = \frac{1}{2}x - 4$.*

Answer: 2

Solution: The slope of the tangent line at the point (x, y) on the curve is $m = f'(x) = -2x + 2$.

If the tangent line is perpendicular to the line $y = \frac{1}{2}x - 4$, we need the slope of the tangent line to be $m = -\frac{1}{\frac{1}{2}} = -2$.

Set up the equation: $-2x + 2 = -2$. Then, solve this equation to obtain $x = 2$.

Therefore, the tangent line of the graph of $y = -x^2 + 2x - 3$ is perpendicular to the line $y = \frac{1}{2}x - 4$ at the point where $x = 2$.

Problem 321. *Let $n \in \mathbb{N}$ be fixed. Suppose that $f^{(k)}(0) = 1$ and $g^{(k)}(0) = 2^k$ for $k = 0, 1, 2, \ldots, n$. Find $\frac{d^n}{dx^n}(f(x)g(x))\Big|_{x=0}$ when $n = 5$.*

Answer: 3^5

Solution: We can use the Leibniz formula:

$$\frac{d^n}{dx^n}(uv) = \sum_{k=0}^{n}\binom{n}{k}u^{(k)}v^{(n-k)},$$

where $u^{(k)}$ denotes the k-th derivative of u and $v^{(n-k)}$ denotes the $(n-k)$-th derivative of v.

In this case, $u = f(x)$ and $v = g(x)$. We are given that $f^{(k)}(0) = 1$ and $g^{(k)}(0) = 2^k$ for $k = 0, 1, 2, \ldots, n$. Substituting these values into the general formula, we get:

$$\frac{d^n}{dx^n}(f(x)g(x))\bigg|_{x=0} = \sum_{k=0}^{n} \binom{n}{k} \cdot 1 \cdot 2^{n-k}.$$

Notice that this sum corresponds to the expansion of $(1 + 2)^n$ according to the binomial theorem. Therefore, we have

$$\frac{d^n}{dx^n}(f(x)g(x))\bigg|_{x=0} = (1 + 2)^n = 3^n.$$

Problem 322. *The function $f(x)$ is defined by*

$$f(x) = \begin{cases} |x|^\alpha \sin(\frac{1}{x}), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

where α is a constant. Find the value of a such that for all $\alpha > a$, the function $f(x)$ is continuous at $x = 0$.

Answer: 0
Solution: Noting that $f(0) = 0$, in order to obtain the continuity of $f(x)$ at $x = 0$ we need

$$\lim_{x\to} f(x) = 0,$$

that is,

$$\lim_{x\to 0} |x|^\alpha \sin \frac{1}{x} = 0.$$

Noting that $\left||x|^\alpha \sin \frac{1}{x}\right| \leq |x|^\alpha$, if $\alpha > 0$, then we have $\lim_{x\to 0} |x|^\alpha = 0$ which implies $\lim_{x\to 0} |x|^\alpha \sin \frac{1}{x} = 0$.

If $\alpha = 0$, $\lim_{x\to 0} |x|^\alpha \sin \frac{1}{x} = \lim_{x\to 0} \sin \frac{1}{x}$ does not exist.

If $\alpha < 0$, we can choose the sequence $x_n = \frac{1}{\frac{\pi}{2} + 2n\pi} \to 0$ as $n \to \infty$ but

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} |x_n|^\alpha \sin\left(\frac{\pi}{2} + 2n\pi\right) = \lim_{n\to\infty} |x_n|^\alpha = +\infty.$$

Therefore, when $\alpha > 0$ the function $f(x)$ is continuous at $x = 0$.

Problem 323. *Evaluate* $\int_0^4 (2x - \sqrt{16 - x^2})dx$.

Answer: 3.43

Solution:
$$\int_0^4 (2x - \sqrt{16 - x^2})dx = \int_0^4 2x\,dx - \int_0^4 \sqrt{16 - x^2}\,dx.$$
For the first integral, we have
$$\int_0^4 2x\,dx = x^2\Big|_0^4 = 4^2 - 0^2 = 16.$$
For the second integral, by a change of variables $x = 4\sin\theta$ we get
$$\begin{aligned}\int_0^4 \sqrt{16 - x^2}\,dx &= \int_0^{\frac{\pi}{2}} \sqrt{16 - 16\sin^2\theta}\;4\cos\theta\,d\theta\\ &= \int_0^{\frac{\pi}{2}} \sqrt{16\cos^2\theta}\;4\cos\theta\,d\theta\\ &= \int_0^{\frac{\pi}{2}} 16\cos^2\theta\,d\theta\\ &= \int_0^{\frac{\pi}{2}} 16\frac{1+\cos(2\theta)}{2}\,d\theta\\ &= 8\int_0^{\frac{\pi}{2}} (1+\cos(2\theta))\,d\theta\\ &= 8\left[\theta + \frac{1}{2}\sin(2\theta)\right]\Bigg|_0^{\frac{\pi}{2}}\\ &= 8\left[\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi\right) - (0+0)\right]\\ &= 4\pi.\end{aligned}$$

So, $\displaystyle\int_0^4 (2x - \sqrt{16 - x^2})\,dx = 16 - 4\pi.$

Problem 324. *Evaluate the series* $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)}$.

Answer: 0.42

Solution: First, express the general term $\frac{1}{(n+1)(n+3)}$ in partial fraction form:
$$\frac{1}{(n+1)(n+3)} = \frac{A}{n+1} + \frac{B}{n+3}.$$
Multiplying both sides by the common denominator $(n+1)(n+3)$ we obtain
$$1 = A(n+3) + B(n+1) \Leftrightarrow 1 = (A+B)n + (3A+B).$$
Thus,
$$\begin{cases} A + B &= 0,\\ 3A + B &= 1.\end{cases}$$

Solving this system of equations, we find that $A = \frac{1}{2}$ and $B = -\frac{1}{2}$.

Now, we have

$$\frac{1}{(n+1)(n+3)} = \frac{1/2}{n+1} - \frac{1/2}{n+3} = \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+3}\right)$$

Now, using the telescoping nature of the series:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)} &= \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+3}\right) \\ &= \frac{1}{2}\left[\left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots\right] \\ &= \frac{1}{2}\left[\frac{1}{2} + \frac{1}{3}\right] = \frac{5}{12}.\end{aligned}$$

Problem 325. *Evaluate the limit* $\lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x}$.

Answer: $-\frac{e}{2}$.
Solution: We can use L'Hôpital's Rule to obtain

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1.$$

Then,

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\ln(1+x)^{\frac{1}{x}}} = \lim_{x\to 0} e^{\frac{\ln(1+x)}{x}} = e^1 = e.$$

Let $f(x)(1+x)^{\frac{1}{x}}$, then $\lim_{x\to 0} f(x) = e$ and the given limit can be written as:

$$\lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x} = \lim_{x\to 0} \frac{f(x) - e}{x}.$$

Now, find the derivative of $f(x)$ by using the chain rule and the quotient rule:

$$\begin{aligned}f'(x) = \frac{d}{dx}(1+x)^{\frac{1}{x}} = \frac{d}{dx}e^{\ln(1+x)^{\frac{1}{x}}} &= \frac{d}{dx}e^{\frac{\ln(1+x)}{x}} \\ &= e^{\frac{\ln(1+x)}{x}} \frac{d}{dx}\frac{\ln(1+x)}{x} \\ &= (1+x)^{\frac{1}{x}} \cdot \frac{\frac{x}{1+x} - \ln(1+x)}{x^2}.\end{aligned}$$

Using L'Hôpital's Rule again to get

$$\begin{aligned}
\lim_{x\to 0}\frac{f(x)-e}{x} = \lim_{x\to 0}\frac{f'(x)}{1} &= \lim_{x\to 0}(1+x)^{\frac{1}{x}}\cdot\lim_{x\to 0}\frac{\frac{x}{1+x}-\ln(1+x)}{x^2}\\
&= e\cdot\lim_{x\to 0}\frac{\frac{(1+x)-x}{(1+x)^2}-\frac{1}{1+x}}{2x}\\
&= e\cdot\lim_{x\to 0}\frac{-1}{2(1+x)^2}\\
&= -\frac{e}{2}.
\end{aligned}$$

Therefore,

$$\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x} = -\frac{e}{2}.$$

Problem 326. *Evaluate the series* $\sum\limits_{n=0}^{\infty}\frac{1}{2n+1}\left(\frac{1}{2}\right)^{2n+1}$.

Answer: $\ln\sqrt{3}$.
Solution: For $x\in(-1,1)$, we have

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}.$$

The series on the right-hand side converges uniformly on any interval $[-x,x]$ for any $x\in(0,1)$. Taking the integrals on both sides yields

$$\int_0^x \frac{1}{1-t^2}dt = \int_0^x\sum_{n=0}^{\infty}t^{2n}dt = \sum_{n=0}^{\infty}\int_0^x t^{2n}dt = \sum_{n=0}^{\infty}\frac{1}{2n+1}x^{2n+1}.$$

Noting that by partial fraction of $\frac{1}{1-t^2} = \frac{1}{2}\left(\frac{1}{1+t}+\frac{1}{1-t}\right)$, we have, for $x\in(0,1)$,

$$\int_0^x \frac{1}{1-t^2}dt = \frac{1}{2}\int_0^x\left(\frac{1}{1+t}+\frac{1}{1-t}\right)dt = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$

So,

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \sum_{n=0}^{\infty}\frac{1}{2n+1}x^{2n+1}.$$

Taking $x=\frac{1}{2}$ leads to

$$\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(\frac{1}{2}\right)^{2n+1} = \frac{1}{2}\ln 3 = \ln\sqrt{3}.$$

Problem 327. *Evaluate the limit* $\lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{\sqrt{n^2-k^2}}$.

Answer: $\frac{\pi}{2}$.
Solution: To evaluate this limit, we can interpret this sum as a Riemann sum and convert it into an integral.

Let $f(x) = \frac{1}{\sqrt{1-x^2}}$ on the interval $[0,1)$. Notice that $f(x)$ is integrable on the interval $[0,1)$.

The given sum can be expressed as:

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{\sqrt{n^2-k^2}} = \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{n} \frac{1}{\sqrt{1-\left(\frac{k}{n}\right)^2}} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right).$$

By the definition of definite integral, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\,dx = \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx.$$

By a substitution of $x = \sin(\theta)$, we have

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx &= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-\sin^2(\theta)}} \cos(\theta)\,d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{\cos(\theta)} \cos(\theta)\,d\theta \\ &= \int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2}\end{aligned}$$

Therefore, we obtain $\lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{\sqrt{n^2-k^2}} = \frac{\pi}{2}$.

An alternative method to evaluate $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$:

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x\Big|_0^1 = \arcsin(1) - \arcsin(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

Problem 328. *Let* α *and* β *be positive constant. If* $\displaystyle\lim_{x\to 0} \frac{1}{\alpha - \cos x} \int_0^x \frac{2t}{\sqrt{\beta + t^2}}\,dt = 1$*, determine the values of* α *and* β*.*

Answer: $\alpha = 1$ and $\beta = 4$.
Solution: Noting that $\displaystyle\lim_{x\to 0} \int_0^x \frac{2t}{\sqrt{\beta+t^2}} dt = 0$, if the given limit exists and equals 1, we must have

$$\lim_{x\to 0} (\alpha - \cos x) = 0.$$

Then, we get $\alpha = 1$.

Using L'Hôpital's rule and the fundamental theorem of calculus, we have

$$\lim_{x\to 0} \frac{1}{1-\cos x} \int_0^x \frac{2t}{\sqrt{\beta+t^2}}dt = \lim_{x\to 0} \frac{\frac{d}{dx}\left(\int_0^x \frac{2t}{\sqrt{\beta+t^2}}dt\right)}{\frac{d}{dx}(1-\cos x)}$$

$$= \lim_{x\to 0} \frac{\frac{2x}{\sqrt{\beta+x^2}}}{\sin x} = 2\lim_{x\to 0} \frac{x}{\sin x} \cdot \lim_{x\to 0} \frac{1}{\sqrt{\beta+x^2}} = 2(1)\left(\frac{1}{\sqrt{\beta}}\right) = \frac{2}{\sqrt{\beta}}.$$

Since this limit equals 1, we must have $\beta = 4$.

Therefore, we obtain $\alpha = 1$ and $\beta = 4$.

Problem 329. *Find the length of the curve of the entire cardioid $r = 1 + \cos\theta$, where the curve is given in polar coordinates.*

Answer: 8.
Solution: We'll use the arc length formula for polar curves:

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

For the cardioid $r = 1+\cos\theta$, we have $\frac{dr}{d\theta} = -\sin\theta$. Now, substitute r and $\frac{dr}{d\theta}$ into the arc length formula and use a change of variables:

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{2+2\cos\theta}\, d\theta = \int_0^{2\pi} 2\left|\cos\left(\frac{\theta}{2}\right)\right| \, d\theta \\ &= \int_0^{\pi} 4\left|\cos(\alpha)\right| \, d\alpha = 8\int_0^{\frac{\pi}{2}} \cos(\alpha)\, d\alpha = 8\sin\alpha\Big|_0^{\frac{\pi}{2}} = 8. \end{aligned}$$

So, the length of the curve for the entire cardioid $r = 1+\cos\theta$ is 8.

Problem 330. *Find the value of the integral* $\int_0^1 \frac{1}{(1+x^2)^2}dx.$

Answer: $\frac{\pi}{8} + \frac{1}{4}$.

Solution: Let $x = \tan\theta$, then $dx = \sec^2\theta\, d\theta$. Substitute these into the integral to obtain

$$\begin{aligned}
\int_0^1 \frac{1}{(1+x^2)^2}\, dx &= \int_0^{\frac{\pi}{4}} \frac{1}{(1+\tan^2\theta)^2} \sec^2\theta\, d\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{1}{(\sec^2\theta)^2} \sec^2\theta\, d\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2\theta}\, d\theta \\
&= \int_0^{\frac{\pi}{4}} \cos^2\theta\, d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{4}} (1+\cos(2\theta))\, d\theta = \frac{1}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]\Bigg|_0^{\frac{\pi}{4}} = \frac{\pi}{8} + \frac{1}{4}.
\end{aligned}$$

Problem 331. *Evaluate the improper integral* $\int_0^\infty \frac{1}{x^2+2x+2} dx.$

Answer: $\frac{\pi}{4}$.
Solution: We can write

$$\int_0^\infty \frac{1}{x^2+2x+2} dx = \int_0^\infty \frac{1}{(x+1)^2+1}\, dx.$$

Now, making the substitution $u = x+1$, so $dx = du$, we have

$$\begin{aligned}
\int_0^\infty \frac{1}{x^2+2x+2} dx &= \int_0^\infty \frac{1}{(x+1)^2+1}\, dx \\
&= \int_1^\infty \frac{1}{u^2+1}\, du \\
&= \lim_{a\to\infty} \int_1^a \frac{1}{u^2+1}\, du \\
&= \lim_{a\to\infty} \arctan(u)\Big|_1^a \\
&= \lim_{a\to\infty} [\arctan(a) - \arctan(1)] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.
\end{aligned}$$

Problem 332. *Find the area of the region outside the circle* $r = 2$ *and inside the cardioid* $r = 2 + 2\cos\theta$*, where the curves are given in polar coordinates.*

Answer: $8 + \pi$.
Solution: The region is bounded by the two curves, so the area A is given by:

$$A = \int_\alpha^\beta \frac{1}{2}\left((2+2\cos\theta)^2 - 2^2\right)\, d\theta.$$

The bounds α and β correspond to the angles at which the two curves intersect. To find these intersection points, set

$$2 = 2 + \cos\theta.$$

Then, $\cos\theta = 0$. For the given two curves, we can take $\theta = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$.

Then, we have $\alpha = -\frac{\pi}{2}$ and $\beta = \frac{\pi}{2}$. Thus,

$$\begin{aligned}
A &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\left((2+2\cos\theta)^2 - 2^2\right) d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(4 + 8\cos\theta + 4\cos^2\theta - 4)\, d\theta \\
&= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (4\cos\theta + 2\cos^2\theta)\, d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos\theta + 1 + \cos(2\theta))\, d\theta \\
&= \left(4\sin\theta + \theta + \frac{1}{2}\sin(2\theta)\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \left(4\sin\left(\frac{\pi}{2}\right) + \frac{\pi}{2} + \frac{1}{2}\sin(\pi)\right) - \left(4\sin\left(-\frac{\pi}{2}\right) - \frac{\pi}{2} + \frac{1}{2}\sin(-\pi)\right) \\
&= \left[4 + \frac{\pi}{2}\right] - \left[-4 - \frac{\pi}{2}\right] \\
&= 8 + \pi.
\end{aligned}$$

So, the area of the region outside the circle $r = 2$ and inside the cardioid $r = 2+2\cos\theta$ is $8+\pi$.

Problem 333. *Evaluate* $\int_0^\infty \frac{1}{1+x^4}\, dx$.

Answer: $\frac{\sqrt{2}\pi}{4}$ or $\frac{\pi}{2\sqrt{2}}$.

Solution: The improper integral $\int_0^\infty \frac{1}{1+x^4}\, dx$ converges. We denote

$$I = \int_0^\infty \frac{1}{1+x^4}\, dx.$$

By changing of variables $x = \frac{1}{y}$ we obtain

$$I = \int_0^\infty \frac{1}{1+x^4}\, dx = \int_0^\infty \frac{y^2}{1+y^4}\, dy = \int_0^\infty \frac{x^2}{1+x^4}\, dx.$$

Then,

$$2I = \int_0^\infty \frac{1}{1+x^4}\, dx + \int_0^\infty \frac{x^2}{1+x^4}\, dx = \int_0^\infty \frac{1+x^2}{1+x^4}\, dx.$$

Hence,

$$\begin{aligned}
I = \frac{1}{2}\int_0^\infty \frac{1+x^2}{1+x^4}\,dx &= \frac{1}{2}\int_0^\infty \frac{1+x^2}{(1+2x^2+x^4)-2x^2}\,dx \\
&= \frac{1}{2}\int_0^\infty \frac{1+x^2}{(1+x^2)^2-2x^2}\,dx \\
&= \frac{1}{4}\int_0^\infty \left[\frac{1}{(1+x^2)+\sqrt{2}x} + \frac{1}{(1+x^2)-\sqrt{2}x}\right]\,dx.
\end{aligned}$$

For $\int_0^\infty \frac{1}{(1+x^2)+\sqrt{2}x}\,dx$, we have

$$\begin{aligned}
\int_0^\infty \frac{1}{(1+x^2)+\sqrt{2}x}\,dx &= \int_0^\infty \frac{1}{\frac{1}{2}+\left(x+\frac{\sqrt{2}}{2}\right)^2}\,dx \\
&= 2\int_0^\infty \frac{1}{1+\left(\sqrt{2}x+1\right)^2}\,dx \\
&= \sqrt{2}\int_1^\infty \frac{1}{1+u^2}\,du \\
&= \sqrt{2}\lim_{a\to\infty}\int_1^a \frac{1}{1+u^2}\,du \\
&= \sqrt{2}\lim_{a\to\infty} \arctan(u)\Big|_1^a \\
&= \sqrt{2}\lim_{a\to\infty}(\arctan(a)-\arctan(1)) \\
&= \sqrt{2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \frac{\sqrt{2}\pi}{4}.
\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
\int_0^\infty \frac{1}{(1+x^2)-\sqrt{2}x}\,dx &= 2\int_0^\infty \frac{1}{1+\left(\sqrt{2}x-1\right)^2}\,dx \\
&= \sqrt{2}\int_{-1}^\infty \frac{1}{1+u^2}\,du \\
&= \sqrt{2}\left(\frac{\pi}{2}-\left(-\frac{\pi}{4}\right)\right) = \frac{3\sqrt{2}\pi}{4}.
\end{aligned}$$

Therefor, $I = \frac{1}{4}\left(\frac{\sqrt{2}\pi}{4}+\frac{3\sqrt{2}\pi}{4}\right) = \frac{\sqrt{2}\pi}{4}$.

Problem 334. *Evaluate the iterated integral* $\int_0^1 dy \int_y^1 (e^{-x^2}+e^x)dx$.

Answer: $\frac{3}{2}-\frac{1}{2}e^{-1}$.
Solution: Noting that the region of the integration is

$$D = \{(x,y) : 0 \le y \le 1, y \le x \le 1\} = \{(x,y) : 0 \le x \le 1, 0 \le y \le x\}$$

and the function $f(x,y) = e^{-x^2} + e^x$ is continuous on D, we have

$$\begin{aligned}
\int_0^1 dy \int_y^1 (e^{-x^2} + e^x)dx &= \iint_D (e^{-x^2} + e^x)dx \\
&= \int_0^1 dx \int_0^x (e^{-x^2} + e^x)dy \\
&= \int_0^1 (e^{-x^2} + e^x)y\Big|_0^x dx \\
&= \int_0^1 (e^{-x^2} + e^x)xdx \\
&= \int_0^1 xe^{-x^2}dx + \int_0^1 xe^x dx.
\end{aligned}$$

By substitution $t = x^2$, we obtain

$$\int_0^1 xe^{-x^2}dx = \frac{1}{2}\int_0^1 e^{-t}dt = -\frac{1}{2}e^{-t}\Big|_0^1 = \frac{1}{2} - \frac{1}{2}e^{-1}.$$

By integration by parts, we have

$$\int_0^1 xe^x dx = \int_0^1 xd(e^x) = xe^x\Big|_0^1 - \int_0^1 e^x dx = e - e^x\Big|_0^1 = e - (e-1) = 1.$$

Combining all the steps, we can obtain

$$\int_0^1 dy \int_y^1 (e^{-x^2} + e^x)dx = \left(\frac{1}{2} - \frac{1}{2}e^{-1}\right) + 1 = \frac{3}{2} - \frac{1}{2}e^{-1}.$$

Problem 335. *Assume that* $a_n > 0$ *for all* $n \in \mathbb{N}$ *and the series* $\sum_{n=1}^{\infty} a_n$ *converges to* 4. *Let* $R_n = \sum_{k=n}^{\infty} a_k$ *for all* $n = 1, 2, \ldots$. *Evaluate* $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{R_n} + \sqrt{R_{n+1}}}$.

Answer: 2.

Solution: Noting that $R_n - R_{n+1} = a_n$ for all n and

$$\frac{a_n}{\sqrt{R_n} + \sqrt{R_{n+1}}} = \frac{a_n}{\sqrt{R_n} + \sqrt{R_{n+}}} \cdot \frac{\sqrt{R_n} - \sqrt{R_{n+1}}}{\sqrt{R_n} - \sqrt{R_{n+1}}} = \frac{a_n(\sqrt{R_n} - \sqrt{R_{n+1}})}{R_n - R_{n+1}} = \sqrt{R_n} - \sqrt{R_{n+1}}.$$

To evaluate the series $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{R_{n+1}} + \sqrt{R_n}}$, we'll use a telescoping series:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{R_n} + \sqrt{R_{n+1}}} &= \sum_{n=1}^{\infty} (\sqrt{R_n} - \sqrt{R_{n+1}}) \\
&= [\sqrt{R_1} - \sqrt{R_2}] + [\sqrt{R_2} - \sqrt{R_3}] + [\sqrt{R_3} - \sqrt{R_4}] + \cdots \\
&= \sqrt{R_1} = \sqrt{\sum_{n=1}^{\infty} a_n} = \sqrt{4} = 2.
\end{aligned}$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{R_n}+\sqrt{R_{n+1}}}$ converges to 2.

Problem 336. *For any $a > 0$ and $b \in \mathbb{R}$, use Sterling's formula*

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{x^x e^{-x}\sqrt{2\pi x}} = 1$$

to evaluate the limit

$$\lim_{n\to\infty} \frac{\Gamma(an+b)}{(n!)^a \, a^{an+b-\frac{1}{2}} n^{b-\frac{1}{2}-\frac{a}{2}} (2\pi)^{\frac{1}{2}-\frac{a}{2}}},$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$ is the gamma function defined for any $\alpha > 0$.

Answer: 1.
Solution: Since $a > 0$, we know that $an + b = (an + b - 1) + 1 \to \infty$ as $n \to \infty$. By Sterling's formula, we have

$$\lim_{n\to\infty} \frac{\Gamma(an+b)}{(an+b-1)^{an+b-1} e^{-(an+b-1)} \sqrt{2\pi(an+b-1)}} = 1.$$

Noting that $\Gamma(n+1) = n!$, we get

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1.$$

Thus,

$$\lim_{n\to\infty} \frac{(n!)^a}{n^{an} e^{-an} (2\pi n)^{\frac{a}{2}}} = 1.$$

Then,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\Gamma(an+b)}{(n!)^a\ a^{an+b-\frac{1}{2}} n^{b-\frac{1}{2}-\frac{a}{2}} (2\pi)^{\frac{1}{2}-\frac{a}{2}}} \\
&= \lim_{n\to\infty} \frac{\Gamma(an+b)}{(an+b-1)^{an+b-1} e^{-(an+b-1)} \sqrt{2\pi(an+b-1)}} \cdot \frac{(an+b-1)^{an+b-1} e^{-(an+b-1)} \sqrt{2\pi(an+b-1)}}{(n!)^a\ a^{an+b-\frac{1}{2}} n^{b-\frac{1}{2}-\frac{a}{2}} (2\pi)^{\frac{1}{2}-\frac{a}{2}}} \\
&= \lim_{n\to\infty} \frac{(an+b-1)^{an+b-1} e^{-(an+b-1)} \sqrt{2\pi(an+b-1)}}{(n!)^a\ a^{an+b-\frac{1}{2}} n^{b-\frac{1}{2}-\frac{a}{2}} (2\pi)^{\frac{1}{2}-\frac{a}{2}}} \\
&= \lim_{n\to\infty} \frac{(an+b-1)^{an+b-1} e^{-(an+b-1)} \sqrt{2\pi(an+b-1)}}{a^{an+b-\frac{1}{2}} n^{b-\frac{1}{2}-\frac{a}{2}} (2\pi)^{\frac{1}{2}-\frac{a}{2}} \cdot n^{an} e^{-an} (2\pi n)^{\frac{a}{2}}} \cdot \frac{n^{an} e^{-an} (2\pi n)^{\frac{a}{2}}}{(n!)^a} \\
&= \lim_{n\to\infty} \frac{(an+b-1)^{an+b-1} e^{-(an+b-1)} \sqrt{2\pi(an+b-1)}}{a^{an+b-\frac{1}{2}} n^{b-\frac{1}{2}-\frac{a}{2}} (2\pi)^{\frac{1}{2}-\frac{a}{2}} \cdot n^{an} e^{-an} (2\pi n)^{\frac{a}{2}}} \\
&= \lim_{n\to\infty} \frac{(an+b-1)^{an+b-\frac{1}{2}} e^{-(b-1)}}{a^{an+b-\frac{1}{2}} n^{b-\frac{1}{2}} n^{an}} \\
&= \lim_{n\to\infty} \frac{(an+b-1)^{an} (an+b-1)^{b-\frac{1}{2}} e^{-(b-1)}}{(an)^{an} n^{b-\frac{1}{2}} a^{b-\frac{1}{2}}} \\
&= \lim_{n\to\infty} \left(\frac{an+b-1}{an}\right)^{an} \left(\frac{an+b-1}{n}\right)^{b-\frac{1}{2}} \frac{e^{-(b-1)}}{a^{b-\frac{1}{2}}} \\
&= \lim_{n\to\infty} \left(1+\frac{\frac{b-1}{a}}{n}\right)^{an} \left(a+\frac{b-1}{n}\right)^{b-\frac{1}{2}} \frac{e^{-(b-1)}}{a^{b-\frac{1}{2}}}.
\end{aligned}$$

Noticing that

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x,\ \forall x \in \mathbb{R},$$

we obtain

$$\lim_{n\to\infty} \left(1+\frac{\frac{b-1}{a}}{n}\right)^{an} = \lim_{n\to\infty} \left[\left(1+\frac{\frac{b-1}{a}}{n}\right)^{n}\right]^a = \left[e^{\frac{b-1}{a}}\right]^a = e^{b-1}.$$

Notice also that $\lim_{n\to\infty} \left(a+\frac{b-1}{n}\right)^{b-\frac{1}{2}} = a^{b-\frac{1}{2}}$.

Therefore, by putting everything together, we can obtain the limit

$$\lim_{n\to\infty} \frac{\Gamma(an+b)}{(n!)^a\ a^{an+b-\frac{1}{2}} n^{b-\frac{1}{2}-\frac{a}{2}} (2\pi)^{\frac{1}{2}-\frac{a}{2}}} = 1.$$

Problem 337. *Consider the differential equation* $\frac{dy}{dx} = xy$. *Find the value of* $y(\sqrt{2})$ *given that* $y(0) = 2$.

Answer: 2e

Solution: First, we solve the differential equation to get $y(x) = 2e^{\frac{1}{2}x^2}$.

$$\frac{dy}{dx} = xy \Leftrightarrow \frac{1}{y}dy = xdx \Leftrightarrow \int \frac{1}{y}dy = \int xdx$$
$$\Rightarrow \ln|y| = \frac{1}{2}x^2 + C \Rightarrow y = \pm e^{\frac{1}{2}x^2+C}.$$

With $y(0) = 2$, we have that $C = \ln 2$ and the solution is $y = 2e^{\frac{1}{2}x^2}$.

Next, we evaluate the function to get $y(\sqrt{2}) = 2e$.

Problem 338. *Solve the following first-order differential equation:*

$$\frac{dy}{dx} + 2y = e^{-x}, \quad y(0) = 1.$$

Answer: $y = e^{-x}$.
Solution: To solve it, we use an integrating factor, $\mu(x) = e^{\int 2dx} = e^{2x}$. Multiplying the entire equation by $\mu(x)$ gives:

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = \frac{d}{dx}(e^{2x}y) = e^{2x}e^{-x} = e^x.$$

Hence, $e^{2x}y = \int e^x dx = e^x + C$, which implies $y = e^{-x} + Ce^{-2x}$.

Using the initial condition $y(0) = 1$, we obtain $1 = y(0) = 1 = e^0 + Ce^{-0} = 1 + C$, so $C = 0$. Therefore, the solution is $y = e^{-x}$.

Problem 339. *Given three vectors* $y_1 = (1, 0, 0)^\top, y_2 = (x, 0, 0)^\top$ *and* $y_3 = (x^2, 0, 0)^\top$. *Does there exist a system of three linear homogeneous ODEs such that all of* y_1, y_2, y_3 *are the solution to this homogeneous ODE system?*

Answer: No.
Solution: Suppose there is such a system. Then, $[y_1, y_2, y_3]$ gives a fundamental matrix of the solution, and $\det[y_1, y_2, y_3] \neq 0$. Consider the linear system $C_1y_1 + C_2y_2 + C_3y_3 = \vec{0}$, which implies $C_1 + C_2x + C_3x^2 = 0$. This quadratic equation cannot hold for all $x \in \mathbb{R}$ unless $C_1 = C_2 = C_3 = 0$, that is, y_1, y_2, y_3 are linearly independent. It implies that the determinant $\det[y_1, y_2, y_3] = 0$, which leads to a contradiction.

Problem 340. *Does the ODE* $x^2y'' + (3x-1)y' + y = 0$ *have a nonzero power series solution near* $x = 0$*?*

Answer: No.
Solution: Assume there exists a power series solution $y = \sum_{n\geq 0} c_nx^n$. Plugging it into the equation, we can get the recursive formula $c_{n+1} = (n+1)c_n$ for $n \geq 0$. Then, $c_n = c_0n!$. But we know $\sum_{n\geq 0} n!x^n$ must be divergent as long as $x \neq 0$.

Problem 341. *Is $y = 0$ a singular solution to $y' = \sqrt{y}\ln(\ln(1 + y))$?*

Answer: Yes.
Solution: First, $y = 0$ is a singular solution to the ODE. In fact, this is a separable ODE, and the solution is given by $x(y) = \int_0^y \frac{dt}{\sqrt{t}\ln\ln(1+t)}$ when $y \neq 0$. On the other hand, $y = 0$ is indeed a solution. So $y = 0$ is a singular solution.

Problem 342. *For the ODE system $x'(t) = y + x(x^2 + y^2)$ and $y'(t) = -x + y(x^2 + y^2)$, is the equilibrium $(x, y) = (0, 0)$ stable?*

Answer: No

Solution: The equilibrium $(x, y) = (0, 0)$ for the linear counterpart is a center, as the coefficient matrix has eigenvalues $\pm i$, purely imaginary. Even if the nonlinearity is locally linear ($o(\sqrt{x^2 + y^2})$ size near $(0, 0)$), we cannot tell the type of the equilibrium $(x, y) = (0, 0)$ for the nonlinear system. Instead, we can introduce the Lyapunov function $V(x, y) = \frac{x^2+y^2}{2}$. Along the trajectory, we compute that

$$\frac{dV}{dt} = xx'(t) + yy'(t) = xy + 2(x^2 + y^2)^2 - xy > 0 \quad \forall (x, y) \neq (0, 0).$$

That is to say, $V(x, y)$ is increasing as t grows. So, any trajectory starting near the origin will penetrate the circles (the trajectories for the linearized system) and leave away from the equilibrium $(x, y) = (0, 0)$. Thus, the equilibrium $(x, y) = (0, 0)$ for the nonlinear system is unstable.

Problem 343. *Assume $x \in \mathbb{R}$ and the function $g(x)$ is continuous and $xg(x) > 0$ whenever $x \neq 0$. For the autonomous ODE $x''(t) + g(x(t)) = 0$, is the equilibrium $x(t) = 0$ stable?*

Answer: Yes

Solution: Let $y = x'$ and we get a ODE system: $x' = y, \quad y' = -g(x)$. We construct the Lyapunov function $V(x, y) := 0.5y^2 + \int_0^x g(t)dt$ which is positive near $(0, 0)$ thanks to $xg(x) > 0$. Then we compute $\partial_t V$ along the trajectory, which is always equal to zero. That is, $\partial_t V$ is non-positive but not negative, so the equilibrium is stable but not asymptotically stable.

Problem 344. *What is the number of limit cycles for the ODE system $x'(t) = -2x+y-2xy^2$ and $y'(t) = y + x^3 - x^2 y$?*

Answer: 0
Solution: Let X, Y be the functions on the right side of the two ODEs. Then, we compute that

$$\partial_x X + \partial_y Y = 1 - x^2 - 2y^2 < 0.$$

Then the limit cycle doesn't exist according to the following lemma: Given a domain $G \subset \mathbb{R}^2$, if there exists a simply-connected domain $D \subset G$ such that $\partial_x X + \partial_y Y$ does not change sign in D and is always nonzero, then there is no periodic solution in D and thus there is no limit cycle. The proof is by contradiction and the usage of Gauss-Green formula.

Problem 345. *Assume $y = y(x, \eta)$ to be the solution to the initial-value problem $y'(x) = \sin(xy)$ with initial data $y(0) = \eta$. Can we assert that $\frac{\partial y}{\partial \eta}(x, \eta)$ is always positive?*

Answer: Yes
Solution: According to the ODE, we have $y(x, \eta) = \eta + \int_0^x \sin(sy(s, \eta))ds$. Take ∂_η and we get $\frac{\partial y}{\partial \eta} = 1 + \int_0^x \cos(sy(s, \eta))s\partial_\eta y ds$. Denote the right side by u and take ∂_x, we get $u' = x\cos(xy)u$, that is $\frac{du}{u} = x\cos(xy)dx$ with $u(0) = 1$. Taking integration, we get

$$\ln u = \int_0^x x\cos(xy)dx \Rightarrow u = \partial_\eta y = \exp(\int_0^x s\cos(sy)ds) > 0.$$

Problem 346. *Does there exists any nonzero function $f(x) \in L^2(\mathbb{R}^n)$ such that f is harmonic in $\mathbb{R}^n$?*

Answer: No.
Solution: If there exists such a function u, then taking Fourier transform, we get $-|\xi|^2\hat{f}(\xi) = 0$. $\hat{f} \in L^2(\mathbb{R}^n)$ and thus it is supported in $\{\xi = 0\}$. So, $\hat{f} = 0$ in L^2 and by Plancherel theorem $f = 0$ in L^2. Since harmonic function is smooth, the function f must be identically zero.

Problem 347. *Let u be a harmonic function in $\mathbb{R}^n$ satisfying $|u(x)| \le 100(100 + \ln(100 + |x|^{100}))$ for any $x \in \mathbb{R}^n$. Can we assert u is a constant?*

Answer: Yes.
Solution: By the gradient estimate for harmonic functions, we have

$$|\nabla u(x)| \le \frac{n}{R} \max_{B(x,R)} |u(x)| \le \frac{100n}{R}(100 + \ln(100 + R^{100})).$$

Let $R \to \infty$ and we get $\nabla u \equiv 0$. So u must be a constant.

Problem 348. *Assume $u(t, x, y)$ solves the wave equation $u_{tt} - u_{xx} - u_{yy} = 0$ for $t > 0, x, y \in \mathbb{R}$ with initial data $u(0, x, y) = 0$ and $u_t(0, x, y) = g(x, y)$ where $g(x, y)$ is a compactly supported smooth function. Find the limit $\lim_{t\to+\infty} t^{1/4}|u(t, x, y)|$ if it exists.*

Answer: 0
Solution: 2D free wave equation has decay rate $O(1/\sqrt{t})$.

Problem 349. *Consider the transport equation $u_t + 2u_x = 0$ for $t > 0, x > 0$ with initial data $u(0, x) = e^{-x}$ for $x > 0$ and boundary condition $u(t, 0) = A + Bt$ for $t > 0$, where A, B are two constants. Find the values of A, B such that there is a solution $u(t, x)$ is C^1 in $\{t \ge 0, x \le 0\}$ to the equation. Present the answer in the form of [A,B].*

Answer: [1,2]

Solution: The general solution to the transport equation is $u(t, x) = F(x - 2t)$. Since $u_0(x) := e^{-x}$ is defined in $\{x > 0\}$, the function $u_0(x - 2t)$ only determines the solution in $\{x > 2t\}$. To determine the solution in $\{0 < x < 2t\}$, we need the boundary data $u(t, 0) = g(t) := A + Bt$. Let $x = 0$ in the general solution, we get $g(t) = F(-t/2)$ for any

$t > 0$. Hence, the solution in $\{0 < x < 2t\}$ is given by $g(t - \frac{x}{2}) = A + B(t - \frac{x}{2})$. To ensure the continuity, we must have $\lim_{t \to 0} g(t) = \lim_{x \to 0} u_0(x)$, which givens $A = e^0 = 1$. To ensure the C^1 differentiability, we must have $\lim_{(t,x) \to 0} u_t + 2u_x = 0$, which gives $g'(0) + 2u_0'(0) = 0$; that is $B = 2$. The solution is

$$u(t,x) = \begin{cases} e^{-x+2t} & x \geq 2t, \\ 1 + 2t - x & 0 \leq x \leq 2t. \end{cases}$$

Problem 350. *In how many ways can you arrange the letters in the word "INTELLIGENCE"?*

Answer: 9979200.
Solution: It is given by the multinomial coefficient $\binom{12}{2,2,1,3,2,1,1} = \frac{12!}{2!2!1!3!2!1!1!} = 9,979,200$.

Problem 351. *Suppose that A, B, and C are mutually independent events and that $P(A) = 0.2$, $P(B) = 0.5$, and $P(C) = 0.8$. Find the probability that exactly two of the three events occur.*

Answer: 0.42.
Solution: $P(A \cap B \cap C) = (0.2)(0.5)(0.8) = 0.08$, $P(A \cap B) = 0.10$, $P(A \cap C) = 0.16$, $P(B \cap C) = 0.40$. $P(A \cap B \cap C') = P(A \cap B) - P(A \cap B \cap C) = 0.02$. Similarly, $P(A \cap B' \cap C) = 0.16 - 0.08 = 0.08$, and $P(A' \cap B \cap C) = 0.40 - 0.08 = 0.32$.
Thus, $P(A \cap B \cap C') + P(A \cap B' \cap C) + P(A' \cap B \cap C) = 0.42$.

Problem 352. *A club with 30 members wants to have a 3-person governing board (president, treature, secretary). In how many ways can this board be chosen if Alex and Jerry don't want to serve together?*

Answer: 24192
Solution: $\binom{2}{1}\binom{28}{2}(3!) + \binom{28}{3}(3!) = 24,192$.

Problem 353. *There are seven pairs of socks in a drawer. Each pair has a different color. You randomly draw one sock at a time until you obtain a matching pair. Let the random variable N be the number of draws. Find the value of n such that $P(N = n)$ is the maximum.*

Answer: 5.
Solution: You absolutely get a matching pair when $n = 8$. For $n = 8$, the first draw can be any sock. The second draw must be one of the 12 that are different, the third draw must be one of the 10 that are different from the first two, ..., the seventh draw must be one of the 2. Thus $P(N = 8) = (12/13)(10/12)(8/11)(6/10)(4/9)(2/8) = 16/429$. Repeat the similar process for $n = 7, 6, \ldots, 2$ to get

$$P(N = 7) = 48/429, P(N = 6) = 80/429, P(N = 5) = 32/143,$$

$$P(N = 4) = 30/143, P(N = 3) = 2/13, P(N = 2) = 1/13.$$

Therefore, $n = 5$ yields the maximum value of $P(N = n)$.

Problem 354. *A pharmacy receives 2/5 of its flu vaccine shipments from Vendor A and the remainder of its shipments from Vendor B. Each shipment contains a very large number of vaccine vials. For Vendor A's shipments, 3% of the vials are ineffective. For Vendor B, 8% of the vials are ineffective. The hospital tests 25 randomly selected vials from a shipment and finds that two vials are ineffective. What is the probability that this shipment came from Vendor A?*

Answer: 0.24

Solution: If the shipment is from Vendor A, the probability that two vials are ineffective is

$$\binom{25}{2}(3\%)^2(97\%)^{23} = 0.134003.$$

If the shipment is from Vendor B, the probability that two vials are ineffective is

$$\binom{25}{2}(8\%)^2(92\%)^{23} = 0.282112.$$

Applying Bayes Theorem, we can obtain the probability that the shipment came from Vendor A give that there are two vials are ineffective in a selected shipment:

$$\frac{(2/5)(0.134003)}{(2/5)(0.134003) + (3/5)(0.282112)} = 0.24051.$$

Problem 355. *Let X_k be the time elapsed between the $(k-1)^{\text{th}}$ accident and the k^{th} accident. Suppose $X_1, X_2, \ldots$ are independent of each other. You use the exponential distribution with probability density function $f(t) = 0.4e^{-0.4t}$, $t > 0$ measured in minutes to model X_k. What is the probability of at least two accidents happening in a five-minute period?*

Answer: 0.59
Solution: The number of accidents in one minute is a Poisson process with mean 0.4. Using the property of the Poisson process, the number of accidents in a five-minute period, denoted by the random variable N, must follow the Poisson distribution with mean $\lambda = (0.4)(5) = 2$.

$$P(N \geq 2) = 1 - P(N = 0) - P(N = 1) = 1 - e^{-2} - 2e^{-2} = 1 - 3e^{-2}.$$

Problem 356. *In modeling the number of claims filed by an individual under an insurance policy during a two-year period, an assumption is made that for all integers $n \geq 0$, $p(n+1) = 0.1p(n)$ where $p(n)$ denotes the probability that there are n claims during the period. Calculate the expected number of claims during the period.*

Answer: 0.11.
Solution: From the given recursive formula, $p(n) = 0.1^n p(0)$ can be derived. Taking into account $\sum_{n=0}^{\infty} p(n) = 1$, we obtain $p(0)\sum_{n=0}^{\infty} 0.1^n = 1$. Solving this equation yields $p(0) = 0.9$. Thus $p(n) = (0.9)(0.1^n)$. This indicates the number of claims follows Geometric distribution, so the expected number of claims is $(1 - 0.9)/0.9 = 0.11$.

Problem 357. *An ant starts at $(1, 1)$ and moves in one-unit independent steps with equal probabilities of 1/4 in each direction: east, south, west, and north. Let W denote the east-west position and S denote the north-south position after n steps. Find $\mathbb{E}[e^{\sqrt{W^2+S^2}}]$ for $n = 3$.*

Answer: 12.08

Solution: We make a shift to assume the ant starts at $(0,0)$, $X = W-1, Y = S-1$. We can find the joint probability function for (X,Y): The four points $(\pm 1, 0)$, $(0, \pm 1)$ each have probability $9/64$, the eight points $(\pm 2, \pm 1)$, $(\pm 1, \pm 2)$ each have probability $3/64$, the four points $(\pm 3, 0)$, $(0, \pm 3)$ each have probability $1/64$. These results can be obtained by counting the paths to the corresponding points. Then $\mathbb{E}[e^{\sqrt{W^2+S^2}}] = \mathbb{E}[e^{\sqrt{(X+1)^2+(Y+1)^2}}] = 12.083$.

Problem 358. *Let the two random variables X and Y have the joint probability density function $f(x,y) = cx(1-y)$ for $0 < y < 1$ and $0 < x < 1-y$, where $c > 0$ is a constant. Compute $P(Y < X | X = 0.25)$.*

Answer: 0.47

Solution: For the joint density function $f(x,y)$, it should satisfy

$$\int_0^1 \int_0^{1-y} f(x,y)dxdy = 1,$$

so the value of the constant c must be 8. The marginal probability density function for X is

$$f_X(x) = \int_0^{1-x} f(x,y)dy = 4x(1-x^2),\ 0 < x < 1.$$

$$P(Y < X|X = 0.25) = \int_0^{0.25} f(y|x = 0.25)dy = \int_0^{0.25} \frac{f(0.25,y)}{f_X(0.25)}dy = \int_0^{0.25} \frac{2(1-y)}{0.9375}dy = 0.46667.$$

Problem 359. *Three random variables X, Y, Z are independent, and their moment generating functions are:*

$$M_X(t) = (1-3t)^{-2.5}, M_Y(t) = (1-3t)^{-4}, M_Z(t) = (1-3t)^{-3.5}.$$

Let $T = X + Y + Z$. Calculate $\mathbb{E}[T^4]$.

Answer: 1389960
Solution: The moment generating function for the random variable T is

$$M_T(t) = M_X(t)M_Y(t)M_Z(t) = (1-3t)^{-10}.$$

Applying the property of moment generating function, we obtain

$$\mathbb{E}[T^4] = M_T^{(4)}(0) = 10 \times 11 \times 12 \times 13 \times 3^4 \times (1-0)^{-14} = 1389960.$$

Problem 360. *The distribution of the random variable N is Poisson with mean Λ. The parameter Λ follows a prior distribution with the probability density function*

$$f_\Lambda(\lambda) = \frac{1}{2}\lambda^2 e^{-\lambda}, \lambda > 0.$$

Given that we have obtained two realizations of N as $N_1 = 1$, $N_2 = 0$, compute the probability that the next realization is greater than 1. (Assume the realizations are independent of each other.)

Answer: 0.37

Solution: We are asked to compute $P(N \geq 1|N_1 = 1, N_2 = 0)$. Taking into account

$$P(N > 1|N_1 = 1, N_2 = 0) = \int_0^\infty P(N > 1|\Lambda = \lambda)f(\lambda|N_1 = 1, N_2 = 0)d\lambda,$$

we will derive the posterior distribution of λ first.

$$f(\lambda|N_1 = 1, N_2 = 0) = \frac{P(N_1 = 1, N_2 = 0|\Lambda = \lambda)f_\Lambda(\lambda)}{\int_0^\infty P(N_1 = 1, N_2 = 0|\Lambda = \lambda)f_\Lambda(\lambda)d\lambda} = \frac{27}{2}\lambda^3 e^{-3\lambda}.$$

Thus,

$$P(N > 1|N_1 = 1, N_2 = 0) = \int_0^\infty (1 - e^{-\lambda} - \lambda e^{-\lambda})\frac{27}{2}\lambda^3 e^{-3\lambda}d\lambda = \frac{47}{128}.$$

Problem 361. *The minimum force required to break a type of brick is normally distributed with mean 195 and variance 16. A random sample of 300 bricks is selected. Estimate the probability that at most 30 of the selected bricks break under a force of 190.*

Answer: 0.70

Solution: The probability that a brick will not be broken under a force of 190 is $P(Z > \frac{190-195}{4}) = 0.8944$. The number of bricks not breaking under a force of 190 follows a Binomial distribution. The probability that at most 30 bricks break is

$$\sum_{n=270}^{300} \binom{300}{n} 0.8944^n 0.1056^{300-n}.$$

This quantity can be approximated by Normal distribution with continuity correction: $P(N > 265.5) = P(Z > \frac{265.5-(300)(0.8944)}{\sqrt{(300)(0.8944)(1-0.8944)}})$. The final answer is 0.7019.

Problem 362. *Find the variance of the random variable X if the cumulative distribution function of X is*

$$F(x) = \begin{cases} 0, & \text{if } x < 1, \\ 1 - 2e^{-x}, & \text{if } x \geq 1. \end{cases}$$

Answer: 0.93

Solution: The random variable X has a point mass at $x = 1$. $P(X = 1) = 1 - 2e^{-1}$.

$$\mathbb{E}[X] = (1)P(X = 1) + \int_1^\infty xf(x)dx = (1 - 2e^{-1}) + \int_1^\infty 2xe^{-x}dx = 1 + 2e^{-1}$$

$$\mathbb{E}[X^2] = (1^2)P(X = 1) + \int_1^\infty x^2 f(x)dx = (1 - 2e^{-1}) + \int_1^\infty 2x^2e^{-x}dx = 1 + 8e^{-1}.$$

$$\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 4e^{-1}(1 - e^{-1}).$$

Problem 363. *The hazard rate function for a continuous random variable X is defined as $h(x) = \frac{f(x)}{1-F(x)}$, where $f(\cdot)$ and $F(\cdot)$ are the probability density function and cumulative distribution function of X respectively. Now you are given $h(x) = 2e^x + 1, x > 0$. Find $P(X > 1)$.*

Answer: 0.01

Solution: Note $h(x) = \frac{F'(x)}{1-F(x)}$. This implies that

$$F(x) = 1 - e^{-\int_0^x h(t)dt} = 1 - e^{-\int_0^x 2e^t+1dt} = 1 - e^{-2e^x-x+2}.$$

Thus, $P(X > 1) = 1 - F(1) = e^{-2e+1} = 0.0118365$.

Problem 364. *Suppose the random variable X has an exponential distribution with mean 1. Find $\min_{x\in\mathbb{R}} \mathbb{E}|X - x|$.*

Answer: 1.69

Solution: Note $\min_{x\in\mathbb{R}} \mathbb{E}|X - x| = \mathbb{E}|X - \pi_{0.5}|$, where $\pi_{0.5} = \ln 2$ is the median of the exponential distribution.

$$\mathbb{E}|X - \ln 2| = \int_0^{\ln 2} (\ln 2 - x)e^{-x}dx + \int_{\ln 2}^\infty (x - \ln 2)e^{-x}dx = 1 + \ln 2.$$

Problem 365. *The joint probability density function for the random variables X and Y is*

$$f(x, y) = 6e^{-(2x+3y)}, \ x > 0, \ y > 0.$$

Calculate the variance of X given that $X > 1$ and $Y > 2$.

Answer: 0.25.
Solution: The marginal density functions can be found as follows.

$$f_X(x) = \int_0^\infty f(x,y)dy = 2e^{-2x},\ x > 0,$$

$$f_Y(y) = \int_0^\infty f(x,y)dx = 3e^{-3y},\ y > 0.$$

Clearly, $f(x,y) = f_X(x)f_Y(y)$ and this implies that the random variables are independent. Thus, $\text{Var}[X|X > 1, Y > 2] = \text{Var}[X|X > 1]$. Taking into account $P(X > 1) = e^{-2}$, we have

$$\mathbb{E}[X|X > 1] = \int_1^\infty 2xe^{-2x} \cdot \frac{1}{e^{-2}}dx = 1.5,$$
$$\mathbb{E}[X^2|X > 1] = \int_1^\infty 2x^2e^{-2x} \cdot \frac{1}{e^{-2}}dx = 2.5.$$

Thus,

$$\text{Var}[X|X > 1, Y > 2] = \text{Var}[X|X > 1] = 2.5 - 1.5^2 = 0.25.$$

Problem 366. *Consider the Markov chain X_n with state space $Z = \{0, 1, 2, 3, \ldots\}$. The transition probabilities are*

$$p(x, x+2) = \frac{1}{2},\ p(x, x-1) = \frac{1}{2},\ x > 0,$$

and $p(0,2) = \frac{1}{2},\ p(0,0) = \frac{1}{2}$. Find the probability of ever reaching state 0 starting at $x = 1$.

Answer: 0.62

Solution: Let $\alpha(x) = P(X_n = 0 \text{ for some } n \geq 0 | X_0 = x)$, then $\alpha(x)$ must satisfy

$$\alpha(x) = p(x, x+2)\alpha(x+2) + p(x, x-1)\alpha(x-1),\ x > 0.$$

To solve the equation

$$\alpha(x) = 0.5\alpha(x+2) + 0.5\alpha(x-1), x > 0$$

with $\alpha(0) = 1$, we set $\alpha(x) = a^x$ and obtain

$$0.5a^3 - a + 0.5 = 0.$$

This cubic equation has three roots

$$a_1 = 1, a_2 = \frac{1}{2}(\sqrt{5} - 1), a_3 = -\frac{1}{2}(\sqrt{5} + 1).$$

Thus, $\alpha(x)$ admits the expression of $c_1 + c_2a_2^x + c_3a_3^x$. By setting $c_1 = 0, c_3 = 0, c_2 = 1$, we can check that $\alpha(x) = \left(\frac{\sqrt{5}-1}{2}\right)^x$ satisfies the properties of a transient Markov chain. Thus, the chain is transit and the probability of ever reaching state 0 starting at x is $\left(\frac{\sqrt{5}-1}{2}\right)^x$.

Problem 367. *The two random variables X and Y are independent and each is uniformly distributed on $[0, a]$, where $a > 0$ is a constant. Calculate the covariance of X and Y given that $X + Y < 0.5a$ when $a^2 = 2.88$.*

Answer: -0.02
Solution: The conditional distribution of X and Y given $X + Y < 0.5a$ must be uniform over the triangular region with vertices $(0, 0)$, $(0, 0.5a)$, $(0.5a, 0)$. Thus,

$$f_{X,Y|X+Y<0.5a}(x, y) = 8a^{-2},\ 0 < x, y < 0.5a,\ x + y < 0.5a.$$

$$\mathbb{E}[X|X + Y < 0.5a] = \int_0^{0.5a} \int_0^{0.5a-x} 8a^{-2}xdydx = \frac{1}{6}a,$$

$$\mathbb{E}[Y|X + Y < 0.5a] = \int_0^{0.5a} \int_0^{0.5a-y} 8a^{-2}ydydx = \frac{1}{6}a,$$

$$\mathbb{E}[XY|X + Y < 0.5a] = \int_0^{0.5a} \int_0^{0.5a-x} 8a^{-2}xydxdy = \frac{1}{48}a^2,$$

$$\mathrm{Cov}[X, Y|X + Y < 0.5a] = \frac{1}{48}a^2 - (\frac{1}{6}a)^2 = -\frac{1}{144}a^2.$$

When $a^2 = 2.88$, we get $\mathrm{Cov}[X, Y|X + Y < 0.5a] = 0.02$.

Problem 368. *There are N balls in two boxes in total. We pick one of the N balls at random and move it to the other box. Repeat this procedure. Calculate the long-run probability that there is one ball in the left box.*

Answer: $N2^{-N}$
Solution: Let X_n be the number of balls in the left box after nth draw. Clearly, X_n is a Markov chain because X_{n+1} just depends on X_n. The transition matrix is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ \frac{1}{N} & 0 & \frac{N-1}{N} & 0 & \dots & 0 & 0 & 0 \\ 0 & \frac{2}{N} & 0 & \frac{N-2}{N} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \frac{N-1}{N} & 0 & \frac{1}{N} \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

Let $\bar{\pi} = (\pi_0, \pi_1, \dots, \pi_N)$ be the stationary distribution. We have $\bar{\pi} = \bar{\pi}P$ that gives the system of equations:

$$\begin{cases} \pi_0 = \frac{1}{N}\pi_1 \\ \pi_1 = \pi_0 + \frac{2}{N}\pi_2 \\ \pi_2 = \frac{N-1}{N}\pi_1 + \frac{3}{N}\pi_3 \\ \dots \\ \pi_{N-1} = \frac{2}{N}\pi_{N-2} + \pi_N \\ \pi_N = \frac{1}{N}\pi_{N-1}. \end{cases}$$

In general, $\pi_K = \frac{N-K+1}{N}\pi_{K-1} + \frac{K+1}{N}\pi_{K+1}$. We can derive that $\pi_K = \binom{N}{K}\pi_0$. Taking into account $\sum_{i=0}^{N}\pi_i = 1$, we can obtain $\pi_0 = 2^{-N}$, and $\pi_K = \binom{N}{K}2^{-N}$ for $K = 0, 1, \ldots, N$.

When $K = 1$, we get $\pi_1 = N2^{-N}$.

Problem 369. *Let W_t be a standard Brownian motion. Find the probability that $W_t = 0$ for some $t \in [1, 3]$.*

Answer: 0.61

Solution: By the reflection principle,

$$
\begin{aligned}
& P(W_t = 0 \text{ for some } t \text{ with } 1 \le t \le 3) \\
= & \int_{-\infty}^{\infty} p_1(0, x) P(W_s = 0 \text{ for some } s \text{ with } 1 \le s \le 3 | W_1 = x) dx \\
= & 2\int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left(2\int_x^{\infty} \frac{1}{\sqrt{4\pi}} e^{-\frac{t^2}{4}} dt\right) dx \\
= & \frac{2}{\pi} \arctan\sqrt{2}.
\end{aligned}
$$

Problem 370. *Consider a random walk on the integers with probability $1/3$ of moving to the right and probability $2/3$ of moving to the left. Let X_n be the number at time n and assume $X_0 = K > 0$. Let T be the first time that the random walk reaches either 0 or $2K$. Compute the probability $P(X_T = 0)$ when $K = 2$.*

Answer: 0.80

Solution: Let $M_n = 2^{X_n}$ and the filtration $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$. We can show that M_n is a martingale with respect to $\mathcal{F}_n$. One can also show that T is finite almost surely and $\mathbb{E}(|M_n|\mathbf{1}_{\{T>n\}}) \to 0$ as $n \to \infty$. By optional sampling theorem, $\mathbb{E}(M_T) = \mathbb{E}(M_0)$. Thus,

$$2^0 P(X_T = 0) + 2^{2K} P(X_T = 2K) = 2^K,$$

and

$$P(X_T = 0) + P(X_T = 2K) = 1.$$

Thus, $P(X_T = 0) = \frac{4^K - 2^K}{4^K - 1}$.

Problem 371. *Given the data set* $\{3, 7, 7, 2, 5\}$*, calculate the sample mean* μ *and the sample standard deviation* σ*. Present the answer as* $[\mu, \sigma]$*.*

Answer: [4.8, 2.28]
Solution: The sample mean $\bar{x}$ is given by $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$. For our data set,

$$\bar{x} = \frac{3+7+7+2+5}{5} = \frac{24}{5} = 4.8.$$

The sample standard deviation s is calculated as $s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$,

$$s = \sqrt{\frac{(3-4.8)^2 + (7-4.8)^2 + (7-4.8)^2 + (2-4.8)^2 + (5-4.8)^2}{4}} \approx 2.28.$$

Problem 372. *A sample of 30 observations yields a sample mean of 50. Assume the population standard deviation is known to be 10. When testing the hypothesis that the population mean is 45 at the 5% significance level, should we accept the hypothesis?*

Answer: No

Solution: We use a Z-test for the hypothesis. The null hypothesis $H_0 : \mu = 45$. The test statistic is

$$Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{50-45}{10/\sqrt{30}} \approx 3.27.$$

At the 5% significance level, the critical value $Z_{0.05} \approx 1.96$. Since $3.27 > 1.96$, we reject H_0.

Problem 373. *Given points* $(1, 2)$*,* $(2, 3)$*,* $(3, 5)$*, what is the slope of the least squares regression line?*

Answer: 1.5

Solution: The least squares regression line is $y = ax + b$ where

$$a = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}, \quad b = \frac{\sum y - a\sum x}{n}.$$

For the given points,

$$a = \frac{3(1\cdot 2 + 2\cdot 3 + 3\cdot 5) - (1+2+3)(2+3+5)}{3(1^2+2^2+3^2) - (1+2+3)^2} = \frac{9}{6} = \frac{3}{2},$$

$$b = \frac{2+3+5-\frac{3}{2}(1+2+3)}{3} = \frac{1}{3}.$$

So, the regression line is $y = \frac{3}{2}x + \frac{1}{3}$.

Problem 374. *A random sample of 150 recent donations at a certain blood bank reveals that 76 were type A blood. Does this suggest that the actual percentage of type A donation differs from 40%, the percentage of the population having type A blood, at a significance level of 0.01?*

Answer: Yes
Solution: We want to test the following hypotheses

$$H_0 : p = 0.4 \quad vs. \quad H_1 : p \neq 0.4.$$

The test statistic is

$$z = \frac{76/150 - 0.4}{\sqrt{0.4 \cdot 0.6/150}} = 2.67.$$

The p-value is

$$2P(Z \geq 2.67) = 0.0076$$

which is smaller than 0.01. So, the data does suggest that the actual percentage of type A donations differs from 40%.

Problem 375. *The accompanying data on cube compressive strength (MPa) of concrete specimens are listed as follows:*

$$112.3 \quad 97.0 \quad 92.7 \quad 86.0 \quad 102.0 \quad 99.2 \quad 95.8 \quad 103.5 \quad 89.0 \quad 86.7.$$

Assume that the compressive strength for this type of concrete is normally distributed. Suppose the concrete will be used for a particular application unless there is strong evidence that the true average strength is less than 100 MPa. Should the concrete be used under significance level 0.05?

Answer: Yes.
Solution: We want to test the following hypotheses

$$H_0 : \mu = 100 \quad vs. \quad H_1 : \mu < 100.$$

The test statistic is

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{96.42 - 100}{8.26/\sqrt{10}} \approx -1.37.$$

The p-value is

$$P(t_9 \leq -1.37) \approx 0.102$$

which is greater than 0.05. So, we do not reject H_0 and so the concrete should be used.

Problem 376. *Suppose we have a sample from normal population as follows.*

$$107.1 \quad 109.5 \quad 107.4 \quad 106.8 \quad 108.1$$

Find the sample mean and sample standard deviation, and construct a 95% confidence interval for the population mean.

Answer: $(106.44, 109.12)$.
Solution: The sample mean is

$$\bar{x} = \frac{107.1 + 109.5 + 107.4 + 106.8 + 108.1}{5} = 107.78$$

and the sample standard deviation is $s = 1.076$. The corresponding 95% confidence interval is

$$\bar{x} \pm t_{0.025,4} s/\sqrt{n} = 107.78 \pm 2.776 \cdot 1.076/\sqrt{5} = (106.44, 109.12).$$

Problem 377. *In a survey of 2000 American adults, 25% said they believed in astrology. Calculate a 99% confidence interval for the proportion of American adults believing in astrology.*

Answer: $(0.225, 0.275)$.
Solution: We have that $n = 2000$ and $\hat{p} = 0.25$. Hence the 99% confidence interval is given by

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.25 \pm 2.576\sqrt{\frac{0.25 \cdot 0.75}{2000}} = 0.25 \pm 0.025 = (0.225, 0.275).$$

Problem 378. *Two new drugs were given to patients with hypertension. The first drug lowered the blood pressure of 16 patients by an average of 11 points, with a standard deviation of 6 points. The second drug lowered the blood pressure of 20 other patients by an average of 12 points, with a standard deviation of 8 points. Determine a 95% confidence interval for the difference in the mean reductions in blood pressure, assuming that the measurements are normally distributed with equal variances.*

Answer: $(-5.9, 3.9)$.
Solution: Note that, for the first sample, we have that $n_1 = 16$, $\bar{x}_1 = 11$ and $s_1 = 6$; and for the second sample, we have that $n_2 = 20$, $\bar{x}_2 = 12$ and $s_2 = 8$. So, the pooled sample variance is

$$s_p^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2} = \frac{15 \cdot 6^2 + 19 \cdot 8^2}{34} \approx 51.647.$$

With $t_{0.05/2, n_1+n_2-2} = t_{0.025,34} \approx 2.03$, the 95% confidence interval for $\mu_1 - \mu_2$ is given by

$$11 - 12 \pm 2.03\sqrt{51.647 \cdot (\frac{1}{16} + \frac{1}{20})} \approx -1 \pm 4.9 \Rightarrow (-5.9, 3.9).$$

Problem 379. *The ages of a random sample of five university professors are 39, 54, 61, 72, and 59. Using this information, find a 99% confidence interval for the population variance of the ages of all professors at the university, assuming that the ages of university professors are normally distributed.*

Answer: $(38.90, 2792.41)$.
Solution: We have that $n = 5$ and the sample variance $s^2 = 144.5$. Meanwhile, the critical values for chi-square distribution with degree of freedom 4 are given by $\chi^2_{0.995,4} = 0.20699$ and $\chi^2_{0.005,4} = 14.8602$. Thus, the 99% confidence interval for the variance is given by

$$\left(\frac{(n-1)s^2}{\chi^2_{0.005,4}}, \frac{(n-1)s^2}{\chi^2_{0.995,4}}\right) = \left(\frac{4 \cdot 144.5}{14.8602}, \frac{4 \cdot 144.5}{0.20699}\right) = (38.90, 2792.41).$$

Problem 380. *Suppose we have two groups of data as follows*

Group 1:	32	37	35	28	41	44	35	31	34
Group 2:	35	31	29	25	34	40	27	32	31

Is there sufficient evidence to indicate a difference in the true means of the two groups at level $\alpha = 0.05$?

Answer: No

Solution: We want to test

$$H_0 : \mu_1 - \mu_2 = 0 \quad vs. \quad H_1 : \mu_1 - \mu_2 \neq 0.$$

Note that, for the first sample, we have that $n_1 = 9$, $\bar{x}_1 = 35.22$ and $s_1^2 = 24.445$; and for the second sample, we have that $n_2 = 9$, $\bar{x}_2 = 31.56$ and $s_2^2 = 20.027$. So, the pooled sample variance is

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2} = \frac{8 \cdot 24.445 + 8 \cdot 20.027}{16} = 22.236,$$

implying the pooled sample standard deviation $s_p = 4.716$.

The test statistic is

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{35.22 - 31.56}{4.716\sqrt{\frac{1}{9} + \frac{1}{9}}} = 1.65.$$

The p-value is given by $2P(t_{16} > 1.65) = 0.1184 > 0.05$, where t_{16} is a t-distribution with degree of freedom 16. Thus, we do not reject H_0 and claim that there is not sufficient evidence to indicate a difference in true mean of two groups.

Problem 381. *Let X be one observation from the pdf*

$$f(x|\theta) = \left(\frac{\theta}{2}\right)^{|x|}(1-\theta)^{1-|x|}, \quad x = -1, 0, 1; \quad 0 \leq \theta \leq 1.$$

Is X a complete statistic?

Answer: No
Solution: Note that

$$E(X) = \frac{\theta}{2} \cdot 1 + (1-\theta) \cdot 0 + \frac{\theta}{2} \cdot (-1) = 0, \quad \forall 0 \leq \theta \leq 1.$$

But X is not equal to 0. By the definition of completeness, X is not a complete statistic.

Problem 382. *Let $X_1, \ldots, X_n$ be an i.i.d. random sample with probability density function (pdf)*

$$f(x|\theta) = \begin{cases} \frac{2}{\sqrt{\pi\theta}} e^{-\frac{x^2}{\theta}}, & x > 0, \\ 0, & \textit{otherwise}; \end{cases}$$

where $\theta > 0$. What is the Cramer-Rao Lower Bound for estimating θ?

Answer: $2\theta^2/n$.
Solution: The likelihood function and log likelihood function are given respectively by

$$L(\theta) = \frac{2^n}{\pi^{n/2}}\theta^{-n/2}e^{-\sum_{i=1}^n X_i^2/\theta},$$

$$\ell(\theta) = n\log(2/\sqrt{\pi}) - \frac{n}{2}\log\theta - \sum_{i=1}^n X_i^2/\theta.$$

Taking the derivatives in θ, we obtain

$$\ell'(\theta) = -\frac{n}{2\theta} + \frac{\sum_{i=1}^n X_i^2}{\theta^2}, \quad \ell''(\theta) = \frac{n}{2\theta^2} - \frac{2\sum_{i=1}^n X_i^2}{\theta^3}.$$

Note that $E(X^2) = \theta/2$, we have the Fisher information

$$I_n(\theta) = -E(\ell''(\theta)) = -\frac{n}{2\theta^2} + \frac{2nE(X^2)}{\theta^3} = \frac{n}{2\theta^2}.$$

Therefore, the Cramer-Rao Lower Bound is given by $1/I_n(\theta) = 2\theta^2/n$.

Problem 383. *Let $X_1, X_2, \ldots, X_n$ be an i.i.d. random sample from the population density (i.e., Exp($\frac{1}{\theta}$))*

$$f(x|\theta) = \begin{cases} \theta e^{-\theta x}, & x > 0; \\ 0, & \text{otherwise.} \end{cases} \qquad \text{where } \theta > 0.$$

Let $\hat{\theta}_n$ be the maximal likelihood estimator of θ. What is the variance of the asymptotic distribution of the limiting distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$?

Answer: θ^2
Solution: Note that $E(X_i) = \frac{1}{\theta}$ and $\text{Var}(X_i) = \frac{1}{\theta^2}$. By the Central Limit Theorem,

$$\sqrt{n}\left(\bar{X}_n - \frac{1}{\theta}\right) \xrightarrow{d} N\left(0, \frac{1}{\theta^2}\right).$$

Note that the likelihood function and log-likelihood function are given respectively by

$$L(\theta) = \theta^n e^{-\theta \sum_{i=1}^n x_i}, \quad \ell(\theta) = n \log \theta - \theta \sum_{i=1}^n x_i.$$

Taking the derivative

$$\ell'(\theta) = \frac{n}{\theta} - \theta \sum_{i=1}^n x_i = 0$$

gives that the MLE is

$$\hat{\theta}_n = \frac{n}{\sum_{i=1}^n X_i} = \frac{1}{\bar{X}_n}.$$

Let $g(t) = \frac{1}{t}$ with $g'(t) = -\frac{1}{t^2}$. By the Delta method, we have

$$\sqrt{n}(\hat{\theta}_n - \theta) = \sqrt{n}(g(\bar{X}_n) - g(\frac{1}{\theta})) \xrightarrow{d} N\left(0, \frac{(g'(\frac{1}{\theta}))^2}{\theta^2}\right) = N(0, \theta^2).$$

Problem 384. *Let $U_1, U_2, \ldots,$ be i.i.d.* Uniform$(0,1)$ *random variables and let $X_n = \left(\prod_{k=1}^n U_k\right)^{-1/n}$. What is the variance of the asymptotic distribution of $\frac{\sqrt{n}(X_n - e)}{e}$ as $n \to \infty$?*

Answer: 1

Solution: Let $Y_n = \log X_n = \frac{1}{n}\sum_{k=1}^{n}(-\log U_k)$. Note that $-\log U_k$ are i.i.d. with Exponential distribution with parameter 1, having mean $\mu = 1$ and variance $\sigma^2 = 1$. By the central limit theorem,

$$\frac{\sqrt{n}(Y_n - \mu)}{\sigma} = \sqrt{n}(Y_n - 1) \xrightarrow{d} N(0, 1).$$

Applying the Delta method with $g(y) = e^y$ such that $g(1) = e$ and $g'(1) = e > 0$, we obtain

$$\sqrt{n}(g(Y_n) - g(1)) \xrightarrow{d} N(0, [g'(1)]^2),$$

which is equivalent to $\sqrt{n}(X_n - e) \xrightarrow{d} N(0, e^2)$, yielding

$$\frac{\sqrt{n}(X_n - e)}{e} \xrightarrow{d} N(0, 1).$$

Problem 385. *Let X be a single observation from* Unifrom$(0, \theta)$ *with density $f(x|\theta) = 1/\theta I(0 < x < \theta)$, where $\theta > 0$. Does there exist Cramer-Rao Lower Bound for estimating θ?*

Answer: No
Solution: Let h be a nonzero function. The existence of Cramer-Rao Lower Bound requires that

$$\frac{d}{d\theta}E_\theta(h(X)) = \int_0^\theta \frac{d}{d\theta}(h(x)f(x|\theta))dx.$$

However, we have that

$$\frac{d}{d\theta}E_\theta(h(X)) = \frac{d}{d\theta}\left(\int_0^\theta h(x)\frac{1}{\theta}dx\right) = \frac{h(\theta)}{\theta} - \frac{1}{\theta^2}\int_0^\theta h(x)dx$$

and

$$\int_0^\theta \frac{d}{d\theta}(h(x)f(x|\theta))dx = -\frac{1}{\theta^2}\int_0^\theta h(x)dx,$$

which are not equal when h is a nonzero function. Thus, the condition for the existence of Cramer-Rao Lower Bound is not satisfied.

In fact, if the Cramer-Rao Lower Bound exists, then it would be given by

$$\frac{1}{E\left(\left(\frac{d}{d\theta}\log f(X|\theta)\right)^2\right)} = \theta^2.$$

However, $2X$ is an unbiased estimator of θ with variance $\theta^2/3$ which is smaller than θ^2, making a contradiction.

Problem 386. *Let $X_1, \ldots, X_n$ be i.i.d. sample from* Gamma(α, β) *with density function $f(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta}$, $x > 0$, $\alpha, \beta > 0$, where α is known and β is unknown. What is the value of the uniform minimum variance unbiased estimator (UMVUE) for $1/\beta$ when $n\alpha = 1$?*

Answer: 0

Solution: As an exponential family, we have that $T = \sum_{i=1}^n X_i$ is a complete and sufficient estimator for β. On the other hand, note that T has a Gamma distribution Gamma$(n\alpha, \beta)$, implying that

$$E\left(\frac{1}{T}\right) = \int_0^\infty \frac{1}{\Gamma(n\alpha)\beta^{n\alpha}} t^{n\alpha-2} e^{-t/\beta} dt = \frac{\Gamma(n\alpha-1)\beta^{n\alpha-1}}{\Gamma(n\alpha)\beta^{n\alpha}} = \frac{1}{(n\alpha-1)\beta}.$$

This shows that $\frac{n\alpha-1}{\sum_{i=1}^n X_i}$ is an unbiased estimator for $1/\beta$. Finally, since $\frac{n\alpha-1}{\sum_{i=1}^n X_i}$ is an estimator based on the complete and sufficient statistic $\sum_{i=1}^n X_i$, by the Lehmann-Scheffé Theorem, $\frac{n\alpha-1}{\sum_{i=1}^n X_i}$ is the UMVUE for $1/\beta$.

Problem 387. *Let $X_1, X_2, \ldots, X_n$ be i.i.d. sample from the population density*

$$f(x|\theta) = \frac{2}{\theta} x e^{-x^2/\theta} I(x > 0), \quad \theta > 0.$$

Consider using appropriate chi-square distribution to find the size α uniformly most powerful (UMP) test for $H_0 : \theta \le \theta_0$ vs. $H_1 : \theta > \theta_0$. Let $\chi^2_{2n,\alpha}$ is the value such that $P(\chi^2_{2n} > \chi^2_{2n,\alpha}) = \alpha$ and χ^2_{2n} is the chi-squared distribution with degree of freedom $2n$. Should the UMP test reject H_0 if $\sum_{i=1}^n X_i^2 > \frac{\theta_0}{2}\chi^2_{2n,\alpha}$?

Answer: Yes
Solution: For $\theta_2 > \theta_1$,

$$\frac{f(x_1, \ldots, x_n|\theta_2)}{f(x_1, \ldots, x_n|\theta_1)} = \frac{\frac{2^2}{\theta_2^n}\left(\prod_{i=1}^n x_i\right) e^{-\sum_{i=1}^n x_i^2/\theta_2}}{\frac{2^2}{\theta_1^n}\left(\prod_{i=1}^n x_i\right) e^{-\sum_{i=1}^n x_i^2/\theta_1}} = \left(\frac{\theta_1}{\theta_2}\right)^n e^{-\sum_{i=1}^n x_i^2\left(\frac{1}{\theta_2}-\frac{1}{\theta_1}\right)},$$

which is increasing in $\sum_{i=1}^n x_i^2$. By the Karlin-Rubin theorem, the size-α UMP test reject H_0 if $\sum_{i=1}^n X_i^2 > c$ where c is some constant such that $P_{\theta_0}(\sum_{i=1}^n X_i^2 > c) = \alpha$.

Note that X_i^2 has the exponential distribution Exp(θ), implying $\sum_{i=1}^n X_i^2$ has the gamma distribution Gamma(n, θ). Thus, $2\sum_{i=1}^n X_i^2/\theta$ has the gamma distribution Gamma$(n, 2)$ which is the same as χ^2_{2n}, the chi-squared distribution with degree of freedom $2n$. Therefore, we have

$$\alpha = P_{\theta_0}\left(\sum_{i=1}^n X_i^2 > c\right) = P(\chi^2_{2n} > 2c/\theta_0),$$

implying that $2c/\theta_0 = \chi^2_{2n,\alpha}$ and hence $c = \frac{\theta_0}{2}\chi^2_{2n,\alpha}$.